

Myopia and Discounting*

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Abstract

We study perfectly patient agents who estimate the value of future events by generating noisy, unbiased simulations and combining those signals with priors to form posteriors. These posterior expectations exhibit *as-if discounting*: agents make choices as if they were maximizing a stream of known utils weighted by a discount function, $D(t)$. This as-if discount function reflects the fact that estimated future utils are a combination of signals and priors, so average expectations are optimally shaded toward the mean of the prior distribution, generating behavior that partially mimics the properties of classical time preferences. When the simulation noise has variance that is linear in the event’s horizon, the as-if discount function is hyperbolic, $D(t) = 1/(1 + \alpha t)$. Our analysis includes a stripped-down Bayesian base case and two complementary, psychologically-enriched extensions: (i) the agent also uses the value of current rewards as an imperfect proxy for future rewards; (ii) the agent aggregates “building blocks” to construct a cognitively costly representation of the future, with predictive accuracy that improves as the number of building blocks increases. Our agents exhibit systematic preference reversals, but have no taste for commitment because they suffer from imperfect foresight, which is not a self-control problem. In our framework, agents exhibit less discounting if they have more domain-relevant experience, are more intelligent, or are encouraged to spend more time thinking about an intertemporal tradeoff. Agents who are unable to think carefully about an intertemporal tradeoff – e.g., due to cognitive load – exhibit more discounting. More myopia tends to coincide with more projection bias. In our framework, patience is unstable, fluctuating situationally with the accuracy of forecasting.

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1 Introduction

Most people appear to act *as if* they have a strong preference for earlier rewards over later rewards. For the last century, economists have usually assumed that this type of behavior reflects (fundamental) time preferences, which economists model with discount factors that multiplicatively weight utils. If the one-period-ahead discount factor is δ , then δ utils experienced now are as valuable as one util experienced next period. If $\delta < 1$, economic agents strictly prefer a current util to a delayed util.

However, such time preferences are only one of many ways to explain the empirical regularity that intertemporal choices are characterized by *declining sensitivity as utils are moved further away in time*. Diminishing sensitivity to future utils is also explained by imperfect information. For example, von Böhm-Bawerk (1890) writes that “we possess inadequate power to imagine and to abstract, or that we are not willing to put forth the necessary effort, but in any event we limn a more or less incomplete picture of our future wants and especially of the remotely distant ones. And then, there are all of those wants that never come to mind at all.” Pigou (1920) similarly observes “that our telescopic faculty is defective, and that we, therefore, see future pleasures, as it were, on a diminished scale. That this is the right explanation is proved by the fact that exactly the same diminution is experienced when, apart from our tendency to forget ungratifying incidents, we contemplate the past.”¹ Pigou believed that our imperfect ability to forecast the future mirrors our imperfect ability to recall the past.

To gain intuition for the role of imperfect forecasting, consider a driver who sees an upcoming pothole and estimates that it is small. A few moments later, the driver realizes that the pothole is very large, but it is too late to avoid hitting it. Striking the large pothole is a reflection of imperfect foresight, not procrastination or laziness. In this case, large delayed consequences are misperceived by an imperfectly farsighted driver. We wouldn’t infer that the driver didn’t care about the impending impact because it was in the “future.” If the driver had foreseen the consequences, they would have braked earlier. It is tautological that people will not respond with perfect foresight to future consequences that they do not fully foresee.

Likewise, consider a sailor who sees a few clouds forming on the horizon and doesn’t immediately take the costly action of charting a new course. When the sailor’s vessel is lashed by a violent storm the next day, it is possible that a bad forecast (not laziness) is the explanation for the earlier inaction.

Decision-making is rife with situations in which a current action/inaction causes a stream of current and future consequences, including myriad consequences that are hard to foresee. If delayed

¹For a review of the history of theories of discounting see Loewenstein (1992).

consequences are typically harder to foresee than immediate consequences, then patient decision-makers might appear to be relatively impatient to an observer (or domain expert) who better foresees the delayed consequences.

The role of imperfect information is also apparent in the seemingly impatient behavior of non-human animals. When monkeys are given an abstract, intertemporal choice task on a computer, they act as if they discount delayed rewards at the rate of 10% *per second*. When the same monkeys are given a temporally analogous foraging task (also presented on a computer screen), the monkeys show very little discounting (Blanchard and Hayden, 2015). Animal behavior appears to be impatient in completely novel domains and patient in domains that are evolutionarily relevant. As Blanchard and Hayden (2015) conclude, “Seemingly impulsive behavior in animals is an artifact of their difficulty understanding the structure of intertemporal choice tasks.”

In the current paper, we argue that behavior arising from imperfect foresight is hard to distinguish from behavior arising from time preferences. To explore this, we study a baseline model with two complementary extensions that enrich the psychological realism of the base case and make additional predictions. The baseline model is a Bayesian decision-maker with *perfectly* patient time preferences who receives noisy signals about the future. The first extension (Section 4.1) assumes that this perfectly patient Bayesian agent also uses proxy value representations of the future (e.g., using a relatively cognitively accessible current value of a reward to estimate a more opaque future value). The second extension (Section 4.2) assumes that agents exert cognitive effort to generate constructive representations of the future by mentally aggregating the key building blocks of a future scenario; the accuracy (and cost) of the mental simulation increases with the number of building blocks.

These different micro-foundations tend to make the same predictions about decision-making, but a few subtle differences arise; see Section 4. For example, consider mental operations that use a (cognitively accessible) current value as a starting point for estimating opaque future values. This specific micro-foundation implies that if a good’s value is perceived to be relatively persistent over time, then its current value will be a relatively good proxy for future value; hence value-persistence will lead to better forecasts of future values (and more sensitivity in decision-making to the perfect foresight value of future goods).

In all of these complementary frameworks, an agent’s forecasting problem leads the agent to behave in a way that is easy to misinterpret as a time preference; we call this seemingly impatient behavior *as-if discounting*. Our analysis shows that lack of perfect foresight generates behavior that has most of the same characteristics of behavior that arise from classical discounting of future rewards. Specifically, we show that a perfectly patient Bayesian decision-maker who receives noisy signals about the future will behave as if she values future utils strictly less than current utils.

Ophthalmic myopia arises when people cannot clearly see distant objects. But myopia also means a “lack of foresight or discernment.”² Such forecasting limitations matter when agents need to judge the value of events that will occur at a temporal distance. In this paper, we show that imperfect foresight – i.e., myopia – generates as-if discounting, even when the actor’s true preferences are perfectly patient. More generally, we show that imperfect foresight makes agents appear to behave more impatiently than implied by their deep time preferences.

Our formal model assumes that agents receive noisy, unbiased signals about future events and combine these signals with their priors to generate posterior beliefs about future events. Our key assumption is that the forecasting noise increases with the horizon of the forecast. We give special attention to the case in which the variance of the forecasting noise rises linearly with the forecasting horizon.

We provide an illustrative example of our framework in Section 2, where we study a binary choice problem: an actor chooses between an early reward and a mutually exclusive later reward. We show that when the variance of forecasting noise rises linearly with the event horizon, Bayesian agents will act as if they are hyperbolic discounters, even though their deep time preferences are perfectly patient.

In Section 3, we describe the broader implications of our framework, and identify predictions that distinguish our framework from time preference models. First, we show that our (perfectly patient) agents exhibit preference reversals of the same kind that are exhibited by agents with hyperbolic and present-biased discount functions. However, these preference reversals *do not* reflect a self-control problem in our model. The preference reversals arise because the agents obtain less noisy information with the passage of time. Accordingly, our agents do not wish to commit themselves; they act the same way as naive hyperbolic discounters (Strotz, 1955; Akerlof, 1991; O’Donoghue and Rabin, 1999a,b) rather than sophisticated ones Laibson (1997).

In the cross-section, our framework implies that agents with greater intelligence exhibit less as-if discounting—their superior forecasting ability enables them to make choices that are more responsive to future utility flows.³

Our framework discusses a psychological mechanism that jointly produces the appearance of both myopia and projection bias (Loewenstein et al. (2003)). In our setting, people with imperfect information about the future use the value of a present reward as an input to their Bayesian forecast of the value of a future experience of the same reward. As a result, the value of a present reward affects the predicted value of the future reward, reflecting a bias from the point of view of an observer who presupposes that the agent has full information.

²Merriam-Webster.

³When we refer to intelligence, we describe a rich set of cognitive capabilities that derive from environmental and genetic variation and their interaction.

In addition, our agents exhibit as-if discounting that is domain-specific. They exhibit less as-if discounting (i) when they have more overall life experience, (ii) when they are more experienced in the specific choice domain, (iii) when they have more time to think about an intertemporal choice (e.g., Imas, Kuhn and Mironova, Forthcoming), and (iv) when they have more cognitive bandwidth to think about their choice (e.g., Benjamin et al., 2013). Other authors, building on an earlier version of this paper,⁴ have developed two more insights. There is less as-if discounting (v) when agents receive specific training in simulating the future (John and Orkin, 2022; Ashraf et al., 2022) and (vi) when agents face high-stakes decisions (Gershman and Bhui, 2020).

We emphasize that the myopia mechanisms that we discuss in this paper co-exist with other intertemporal choice phenomena, such as classical (exponential) discounting, present bias, framing effects, demand effects, and sub-additivity.⁵ We set up our model without these other intertemporal choice mechanisms to isolate and highlight the effects of imperfect foresight. Our framework complements other psychological mechanisms (including classical discounting) that also lead agents to respond less to variation in future rewards than to variation in current rewards.

Despite this similarity, our analysis offers specific insights that differentiate myopia from other mechanisms that generate discounting. Most importantly, discounting arising from myopia does not have normative import. Myopic discounting would be mitigated if agents were better informed about the future and vanish entirely if agents were fully informed about the future. Accordingly, our framework explains why agents who have deep preferences that are patient might act in ways that appear to be less patient to an observer. Hence, our framework creates a wedge between revealed preferences and normative true preferences, with the former being (generally) more impatient. This wedge is generated by imperfect foresight (myopia). Our framework also identifies a host of mechanisms that can and should be deployed to enable people to better forecast the future and, as predicted by our framework, thereby act more patiently. For example, Liebman and Luttmer (2015) show that an educational intervention fielded for older workers highlighting the benefits of deferring Social Security initiation leads to a four percentage point reduction in early retirement (one year after the intervention). Our framework explains why cognitive interventions are likely to make people behave more patiently and why this is normatively desirable (see also John and Orkin (2022); Ashraf et al. (2022)). Our theory also explains why particular environmental factors (like being forced to think harder about a decision) also lead to more patient choices.

In Section 5, we generalize our discrete-choice examples by making the action set continuous. We

⁴See the 2017 NBER working paper version of this paper, Gabaix and Laibson (2017)

⁵For example, sub-additivity (Read (2001) has recently been modeled by Enke and Graeber (2021), who microfound this empirical regularity as cognitive uncertainty about the horizon (coupled with mental framing that deploys differencing horizons), which is related to but different from our assumption of uncertainty about the value of a good. We believe that their model provides a useful explanation of sub-additivity. Another microfoundation is that agents require a premium for any delay of reward.

provide sufficient conditions that imply that perfectly patient agents who are imperfect forecasters will act as if they are naive hyperbolic discounters. In Section 6, we discuss connections between our framework and related literatures on myopia, Bayesian cognition, risk, and discounting. Section 7 concludes.

2 A Basic Case: Binary Choice

We introduce our approach by describing a simple example of a binary choice. Consider an agent at time zero, who must choose (irreversibly) between two mutually exclusive rewards: *Early* and *Late*. Reward *Early* would be experienced at date, $t \geq 0$. Reward *Late* would be experienced at date, $t + \tau > t$ (i.e., $\tau > 0$). The agent doesn't know the *true* value of *Early* and *Late*, respectively denoted u_t and $u_{t+\tau}$. To simplify exposition and without loss of generality, we assume that these utility events are deterministic, though they were originally generated from a prior distribution that we will characterize below. (Note that any non-deterministic, zero-mean component is irrelevant because we are operating in utility space and we assume that our agents have classical expected utility preferences.)

Although the agent doesn't know the values of u_t and $u_{t+\tau}$, the agent can mentally simulate these deterministic rewards and thereby generate unbiased signals of their value:

$$\begin{aligned}s_t &= u_t + \varepsilon_t \\ s_{t+\tau} &= u_{t+\tau} + \varepsilon_{t+\tau}.\end{aligned}$$

In the first equation, u_t is the true value of the *Early* utils and ε_t is the simulation noise. In the second equation, $u_{t+\tau}$ is the true value of the *Late* utils and $\varepsilon_{t+\tau}$ is the associated simulation noise. For tractability, we assume that the simulation noise is Gaussian. To simplify exposition, we assume that the correlation between ε_t and $\varepsilon_{t+\tau}$ is zero.⁶

⁶Here we suppose that the decision is made at time 0. If it is made (or revised) at decision time $d \leq t$, our analysis remains the same under the following benchmark case. Suppose that at the decision time, d , the agent contemplating consumption at time $t \geq d$ receives a signal $s_{d,t} = u_t + \sum_{k=d}^t \eta_{k,t}$ where $\eta_{k,t}$ are jointly independent, mean zero Gaussian variables with variance σ_{t-k}^2 (so that when $d = 0$, then $\varepsilon_t = \sum_{k=0}^t \eta_{k,t}$). This implies that the signal received at time d is a sufficient statistic for the decision that the agent faces. In particular, the history of signals received before d does not add incremental information to the signal received at d . Hence, our analysis goes through independently of the quality of the memory of our agent: only the current signal matters for the relevant decision.

2.1 Simulation Noise

We assume that the longer the horizon, the greater the variance of the simulation noise. Intuitively, the further away the event, the harder it is to accurately simulate the event's utility. Because our set-up assumes that $t < t + \tau$, this assumption implies that

$$\text{var}(\varepsilon_t) < \text{var}(\varepsilon_{t+\tau}). \quad (1)$$

We will also sometimes assume that $\lim_{t \rightarrow \infty} \text{var}(\varepsilon_t) = \infty$; however this property is not necessary for our qualitative results.

We will pay particular attention to the special case of simulation noise that has a variance that is proportional to the simulation horizon (and hence, noise that vanishes for events that occur in the present):

$$\text{var}(\varepsilon_t) = \sigma_{\varepsilon_t}^2 = \sigma_{\varepsilon}^2 t \quad (2)$$

$$\text{var}(\varepsilon_{t+\tau}) = \sigma_{\varepsilon_{t+\tau}}^2 = \sigma_{\varepsilon}^2 (t + \tau). \quad (3)$$

This linearity assumption engenders a specific (hyperbolic) functional form in the analysis that follows. But this linearity assumption is not necessary for our qualitative results. We provide a complete characterization of noise functions below: i.e., necessary and sufficient conditions for the noise function to generate as-if discounting with declining discount rates as the horizon increases. The case of linear variance is a special case in this larger class of noise functions.

2.2 Bayesian Priors and Posteriors

The agents in our model combine Bayesian priors with their signals (s_t and $s_{t+\tau}$) to generate a Bayesian posterior. We model the Bayesian prior over utility events (in whatever class of events we are studying) as a Gaussian density with mean μ and variance σ_u^2 :

$$u \sim \mathcal{N}(\mu, \sigma_u^2). \quad (4)$$

Here μ is the average value in this class of utility events (e.g., visits to Philadelphia), whereas σ_u^2 is the overall variance within the class (e.g., some trips are great-Philadelphia in June—and some trips are much less great-Philadelphia in January).

In the appendix, we derive the agent's Bayesian posterior distribution of u_t , which is generated

by combining her prior (4) and her signal s_t :

$$u_t \sim \mathcal{N} \left(\mu + \frac{s_t - \mu}{1 + \frac{\sigma_{\varepsilon_t}^2}{\sigma_u^2}}, \left(1 - \frac{1}{1 + \frac{\sigma_{\varepsilon_t}^2}{\sigma_u^2}} \right) \sigma_u^2 \right). \quad (5)$$

We summarize these results with the following proposition.

Proposition 1 *If the agent generates a mental simulation s_t , then her Bayesian posterior will be*

$$u_t \sim \mathcal{N} \left(\mu + D(t)(s_t - \mu), (1 - D(t)) \sigma_u^2 \right),$$

where

$$D(t) = \frac{1}{1 + \frac{\sigma_{\varepsilon_t}^2}{\sigma_u^2}}, \quad (6)$$

the variance of her simulation noise is $\sigma_{\varepsilon_t}^2$ and her prior distribution is $u \sim \mathcal{N}(\mu, \sigma_u^2)$.

For reasons that will become apparent below (see Proposition 4 in particular), we refer to $D(t)$ as the agent's as-if discount function. Because we assume that simulation noise, $\sigma_{\varepsilon_t}^2$, is increasing in t , $D(t)$ is decreasing in t , which is a standard property of a discount function. If $\lim_{t \rightarrow \infty} \text{var}(\varepsilon_t) = \infty$, then $\lim_{t \rightarrow \infty} D(t) = 0$, another common property of a discount function. In this case, the posterior expectation of u_t converges to the mean of the prior as the horizon increases. In notation,

$$\lim_{t \rightarrow \infty} \mathbb{E}_0[u_t \mid s_t] = \mu.$$

It is helpful to integrate posteriors over agents in the economy. We assume that the signals s_t are unbiased, so they are equal to u_t on average. Accordingly, the average forecast of u_t will be

$$\int_{s_t} \mathbb{E}_0[u_t \mid s_t] dF(s_t \mid u_t) = \mu + D(t)(u_t - \mu). \quad (7)$$

In general, the mean of the prior will be less extreme than the actual values of u_t . To model this statistical property, consider the illustrative case in which the prior is approximately equal to zero (we will relax this restriction later). Under this restriction, the average belief is

$$\int_{s_t} \mathbb{E}_0[u_t \mid s_t] dF(s_t \mid u_t) = D(t)u_t.$$

We now have an expression that looks like a discounted utility framework: $D(t)$ is a decreasing function, and it multiplies the actual utility value u_t .

2.3 Hyperbolic As-if Discounting

We explore a benchmark case: noise that is zero for immediate events and is linear in the horizon.

Proposition 2 *When we assume that $\text{var}(\varepsilon_t) = \sigma_{\varepsilon_t}^2 = \sigma_\varepsilon^2 t$, we obtain hyperbolic as-if discounting:*

$$D(t) = \frac{1}{1 + \alpha t} \quad (8)$$

where

$$\alpha = \frac{\sigma_\varepsilon^2}{\sigma_u^2}, \quad (9)$$

which is the (one-period) noise-to-signal variance ratio.⁷

The discount function, $D(t) = \frac{1}{1 + \alpha t}$, implies an instantaneous discount rate

$$\text{discount rate} = - \frac{D'(t)}{D(t)} = \frac{\frac{\alpha}{(1 + \alpha t)^2}}{\frac{1}{1 + \alpha t}} = \frac{\alpha}{1 + \alpha t}.$$

At horizon 0, the as-if discount rate is α . The as-if discount rate falls with t . As $t \rightarrow \infty$, the as-if discount rate converges to 0.

2.4 An Example When the Mean Prior Is not Zero ($\mu \neq 0$)

As we noted above, actual utility events will tend to be more extreme than priors. To capture this property, we previously set the mean of the prior distribution equal to zero: $\mu = 0$. We now relax this restriction and illustrate the general case with an example in which the mean of the prior distribution is $\mu = 1$. For this example, we assume that the simulation variance is linear in the time horizon and that the variance ratio is $\frac{\sigma_\varepsilon^2}{\sigma_u^2} = 0.1$. Figure 1 plots the population-level expectations of u_t for three values of u_t (holding the mean of the prior distribution fixed at $\mu = 1$): $u_t \in \{3, 1/2, -1\}$.

When the three utility events are in the present ($t = 0$), the three expectations are equal to the true value of each utility event, respectively 3, 1/2, and -1 . However, as the three utility events recede into the distant future, the three expectations revert to the mean of the prior, $\mu = 1$. This discounting towards the mean of the prior is hyperbolic because we assume a linear variance (see subsection 2.3).

⁷There is a trivial generalization of this proposition. When the variance of the forecasting noise is affine in the time horizon, so that $\text{var}(\varepsilon_t) = \sigma_{\varepsilon_t}^2 = \kappa + \sigma_\varepsilon^2 t$, we also obtain hyperbolic as-if discounting:

$$D(t) = \frac{1}{1 + \theta + \alpha t}, \quad \theta = \frac{\kappa}{\sigma_u^2}.$$

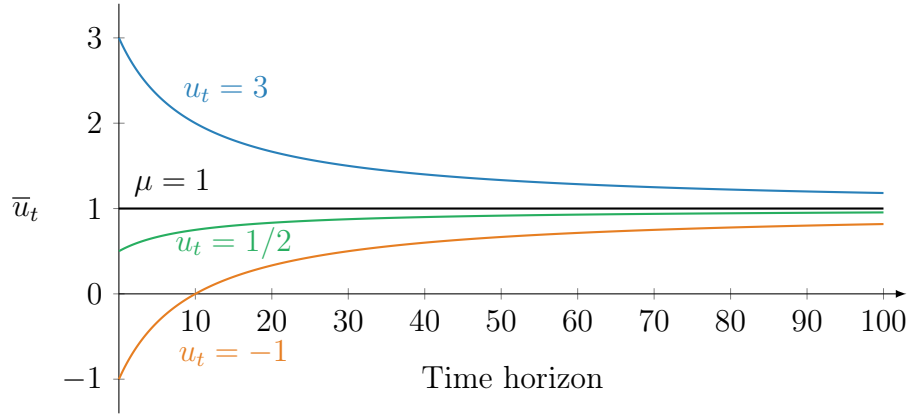


Figure 1: Plot of the average perceived utility $\bar{u}_t$ for three different true utilities $u_t \in \{-1, 1/2, 3\}$, as a function of the time horizon t . This average perceived value is: $\bar{u}_t = \mu + \frac{u_t - \mu}{1 + \frac{\sigma_\varepsilon^2}{\sigma_u^2} t}$ (see (7)). The figure uses $\sigma_\varepsilon^2 / \sigma_u^2 = 0.1$.

The $u_t = 3$ curve is characterized by standard discounting. The further ahead the utility event is shifted, the lower the perceived value of the event. The $u_t = -1$ curve is also characterized by standard discounting on part of its domain. As the event is moved further into the future, its value first declines toward zero. However, at $t = 10$, the perceived value crosses the x -axis and continues converging toward $\mu = 1$. Finally, the $u_t = 1/2$ line displays anti-discounting over the entire domain. The further the value is moved into the future, the higher its perceived value as it converges to the prior mean of $\mu = 1$.

These three lines illustrate the three types of cases that arise in our framework, including the special case of anti-discounting. Note that anti-discounting arises when the true value of u_t lies between 0 and the mean of the prior distribution, μ .⁸

2.5 Another Example: Potholes

Return to the example from the introduction of the paper: a driver who strikes a large pothole that she could have avoided (with perfect foresight). We now map this intuitive example to the notation of our model.

Assume that the driver perceives a choice between (i) adjusting her car's path at an immediate payoff of $u_0 = -2$ when she first imperfectly perceives a *distant* pothole, or (ii) staying on her original course and striking the pothole at a stochastic payoff of u_τ . The Bayesian driver estimates

⁸There are three reasons why the prior, μ , will often be close to 0. First, good and bad hedonic events are both frequent occurrences, so on average $\mu \approx 0$ is a natural benchmark. Second, there are opportunity costs: even pleasant events can have negative net value given some opportunity cost (such as in a job search model, where most job offers should be declined as they have a negative net value once the opportunity cost has been incorporated). Third, suppose that the valuation is experienced as a difference relative to a reference point that is the ex ante value of the object: then by construction $\mu = 0$.

that the expected cost of striking the pothole is

$$\mathbb{E}_0[u_\tau \mid s_\tau] = \mu + \frac{1}{1 + \frac{\sigma_\varepsilon^2}{\sigma_u^2}}(s_\tau - \mu).$$

If almost all potholes are small, then μ is close to zero, and σ_u^2 is close to zero. Because potholes are hard to see from a distance, σ_ε^2 is large. Accordingly, her estimate of the damage from the pothole when she first spies it at a distance will be close to zero (even if s_τ is large in magnitude, i.e. the pothole seems large at first sight). For example, suppose the true damage from the pothole is $u_\tau = -4$. Assume that $\mu = -1$, $\sigma_u^2 = 1$, and $\sigma_\varepsilon^2 = 9$. Then, drivers will on average estimate that striking the pothole will generate a payoff of

$$\mathbb{E}_0[u_\tau \mid s_\tau] = -1 + \frac{1}{1 + \frac{9}{1}}[-4 - (-1)] = -1.3.$$

Accordingly, the driver (on average) chooses to stay the course, leading her to strike the pothole with a payoff of -4 when she could have avoided the pothole (with perfect foresight) at a payoff of -2 . Naturally, staying on her original course is a rational choice given the information that she had when she first saw the pothole at a distance.

2.6 Probabilistic Choice

Our framework implies that choice is probabilistic, because agents receive noisy signals about the value of future rewards. In our example, the agent chooses *Early* if and only if

$$D(t)s_t \geq D(t + \tau)s_{t+\tau},$$

where $D(t)$ is the as-if discount function introduced above and s_t and $s_{t+\tau}$ are the (unbiased) signals of the respective values of the *Early* and *Late* rewards.

From the perspective of an observer who knows the values of u_t and $u_{t+\tau}$, the probability that the agent chooses the *Early* reward is

$$\begin{aligned} \mathbb{P}(\text{choose } \textit{Early}) &= \mathbb{P}[D(t)(u_t + \varepsilon_t) \geq D(t + \tau)(u_{t+\tau} + \varepsilon_{t+\tau})] \\ &= \Phi\left(\frac{1}{\Sigma}[D(t)u_t - D(t + \tau)u_{t+\tau}]\right), \end{aligned} \tag{10}$$

where Φ is a Gaussian CDF and Σ is a scaling factor:

$$\Sigma = \sqrt{D(t)^2 \text{var}(\varepsilon_t) + D(t + \tau)^2 \text{var}(\varepsilon_{t+\tau})}. \tag{11}$$

and for simplicity we assume $\mu = 0$ in this subsection. This restriction is innocuous as long as the two μ values are the same (for the earlier and later rewards).⁹

This probabilistic choice function has natural properties. If $t = 0$ (i.e., the *Early* reward is an immediate reward), then

$$\begin{aligned}\mathbb{P}(\text{choose } \textit{Early}) &= \mathbb{P}[u_0 \geq D(\tau)(u_\tau + \varepsilon_\tau)] \\ &= \Phi\left(\frac{1}{\Sigma}[u_0 - D(\tau)u_\tau]\right).\end{aligned}$$

If we let the time delay between the *Early* reward and the *Late* reward go to infinity (i.e., $\tau \rightarrow \infty$), then

$$\lim_{\tau \rightarrow \infty} \mathbb{P}(\text{choose } \textit{Early}) = \mathbb{1}_{u_0 > 0}.$$

This implies that the agent chooses the *Early* reward with probability one if three properties hold: (i) the *Early* reward is available immediately ($t = 0$), (ii) the *Late* reward is available arbitrarily far in the future ($\tau \rightarrow \infty$), and (iii) the *Early* reward is strictly positive ($u_0 > 0$). In other words, the agent behaves as if she places no value on the (infinitely) delayed *Late* reward.

Now assume that the *Early* reward is available with some delay, so that $t > 0$ (i.e., the *Early* reward is not immediate), then

$$\begin{aligned}\lim_{\tau \rightarrow \infty} \mathbb{P}(\text{choose } \textit{Early}) &= \mathbb{P}[u_t + \varepsilon_t > 0] \\ &= \Phi\left(\frac{u_t}{\Sigma}\right).\end{aligned}$$

Accordingly, if the *Late* reward is available arbitrarily far in the future ($\tau \rightarrow \infty$), then the agent chooses the *Early* reward with the same probability she assigns to the event of the *Early* reward having positive value. Once again, the agent behaves as if she places no value on the (infinitely) delayed *Late* reward.

2.7 Preference Reversals without Commitment

In our setting, an observer who knows the values of u_t and $u_{t+\tau}$ will be able to predict (probabilistic) preference reversals. For example, consider the case of linear variances. In addition, assume that $u_{t+\tau} > u_t > 0$ and that

$$u_t > D(\tau)u_{t+\tau}.$$

⁹Replace u_t by $u_t - \mu$ as needed.

When the two options are sufficiently far in the future (large t), a majority of agents (if forced to choose) will prefer *Late* over *Early* because

$$\mathbb{P}(\text{choose } \textit{Early}) = \Phi\left(\frac{1}{\Sigma} [D(t)u_t - D(t+\tau)u_{t+\tau}]\right) < \frac{1}{2}. \quad (12)$$

To see this, note that

$$D(t)u_t - D(t+\tau)u_{t+\tau} = \frac{u_t}{1+\alpha t} - \frac{u_{t+\tau}}{1+\alpha(t+\tau)}.$$

For sufficiently large values of t , $u_{t+\tau} > u_t$ implies,

$$\frac{u_t}{1+\alpha t} - \frac{u_{t+\tau}}{1+\alpha(t+\tau)} < 0.$$

However, with the passage of time, all agents will eventually choose *Early* because $u_t > D(\tau)u_{t+\tau}$. More precisely, if agents were not forced to choose in advance, but were instead given the chance to choose at time t , all would choose *Early*.

In many economic models, such preference reversals are a sign of dynamic inconsistency in preferences.¹⁰ That is not the case here. The agents in the current model have imperfect information, not dynamically inconsistent time preferences. Their externally predictable preference reversals are a result of their imperfect information. Accordingly, the agents in our model will not desire to limit their own choice sets. Preference reversals arise from their inference problems, not self-control problems.

2.8 More General Discounting Functions

We can provide necessary and sufficient conditions for the as-if discount function, $D(t)$, to exhibit decreasing impatience. In other words, we can derive necessary and sufficient conditions for the property that the instantaneous as-if discount rate

$$\rho(t) := -\frac{D'(t)}{D(t)}$$

is decreasing in the horizon t .

¹⁰See McGuire and Kable (2012, 2013) for a setting in which preference reversals arise because of rational learning dynamics. If a delayed reward that was probabilistically expected does not arrive after a period of waiting, the agent infers that the rate of arrival is low and that further waiting is not likely to pay off and therefore reverts to choosing the immediate reward.

Proposition 3 *The as-if discount function $D(t)$ exhibits strictly decreasing impatience at time horizon t if and only if*

$$\frac{d^2 \sigma_{\varepsilon_t}^2}{dt^2} (\sigma_u^2 + \sigma_{\varepsilon_t}^2) < \frac{d\sigma_{\varepsilon_t}^2}{dt}. \quad (13)$$

This proposition is proved in the appendix. Because we assume that $\frac{d\sigma_{\varepsilon_t}^2}{dt} > 0$, this proposition yields an immediate corollary.

Lemma 1 *The as-if discount function $D(t)$ exhibits strictly decreasing impatience if the variance of simulation noise, $\text{var}(\varepsilon_t) = \sigma_{\varepsilon_t}^2$, is a weakly concave function of time.*

It is natural to assume that the variance of simulation noise, $\text{var}(\varepsilon_t) = \sigma_{\varepsilon_t}^2$, is a weakly concave function of time because many (once) integrated processes have this property, at least asymptotically – e.g. all autoregressive processes of the $\text{AR}(k)$ form. Note too that this concavity property is a sufficient but not necessary, condition for decreasing impatience.

This lemma implies that our model generates as-if discount rates that decrease as the horizon increases for the leading cases of imperfect foresight. We next study a boundary case.

Exponential As-if Discounting Our framework can also be reverse-engineered to generate exponential discounting as a special case. However, this requires assumptions on the variance function that we believe are heroic.

Lemma 2 *The as-if discount function $D(t)$ exhibits a constant discount factor, δ , if and only if*

$$\sigma_{\varepsilon_t}^2 = \left(\frac{1}{\delta^t} - 1 \right) \sigma_u^2.$$

Accordingly, the discount rate is exponential if and only if the simulation variance, $\sigma_{\varepsilon_t}^2$, rises exponentially. This lemma is proven by setting

$$D(t) = \frac{1}{1 + \frac{\sigma_{\varepsilon_t}^2}{\sigma_u^2}} = \delta^t.$$

This sort of exponential cognitive discounting is useful because it captures the idea of cognitive myopia while keeping the tractability of exponential discounting (see Gabaix (2020) for applications to macroeconomics, with a different microfoundation).

3 Implications

We now discuss the key predictions of our model, emphasizing several ways that our model of myopia differs from other models in the intertemporal choice literature. As discussed above, our myopic agent acts as if she is maximizing a utility function with an as-if discount function, $D(\tau)$, where

$$D(\tau) = \frac{1}{1 + \frac{\sigma_{\varepsilon\tau}^2}{\sigma_u^2}}.$$

When the variance of the forecasting noise is weakly concave in the simulation horizon, the discounting function is characterized by an instantaneous discount rate that falls with the horizon. When the forecasting noise is linear in the simulation horizon, so that $\sigma_{\varepsilon\tau}^2 = \sigma_{\varepsilon}^2 \tau$, then the discount function is hyperbolic:

$$D(\tau) = \frac{1}{1 + \frac{\sigma_{\varepsilon}^2}{\sigma_u^2} \tau}.$$

These as-if discounting functions arise because of the imperfect information that the agent has when she generates forecasts. If she were asked to describe her preferences, she would say that she has no time preferences. In other words, she is *trying* to maximize

$$\sum_{\tau=0}^{T-t} u(a_{t+\tau}).$$

Her as-if discounting behavior arises because she doesn't have perfect foresight regarding the future utility flows $u(a_{t+\tau})$.

The next proposition formalizes the sense in which she “appears” to be discounting the future. In this proposition, we introduce a “confused social scientist” who uses the choice data of agents as described in this paper, but the scientist doesn't understand the true structural model. The scientist thinks that the agent chooses according to a true discount function $\hat{D}(t)$, which the scientist is trying to estimate.¹¹ To model noise, the scientist uses a Probit model (i.e., a random choice model with Gaussian distributed noise). The next proposition shows that he will estimate a discount function equal to our (normalized) as-if discount function $D(t)/D(0)$.¹²

Proposition 4 *Consider a confused social scientist modelling the agent described in this paper. The scientist fits a Probit model assuming that the agent has discounted utility $\sum_{t=0}^T \hat{D}(t) \hat{u}_t$ (with*

¹¹We make the simplifying assumption that the scientist knows everything else about the decision environment, including $u(c_t)$.

¹²The normalization is necessary for the special case in which $D(0)$ is not equal to one. This case arises when $\sigma_{\varepsilon_0}^2$ is not equal to zero (in other words, when utils that are going to be *immediately* experienced are still at least partially noisy in their forecasted value).

$\hat{u}_t$ and $\hat{D}(t)$ unknown, normalizing $\hat{D}(0) = 1$). The scientist’s dataset is a collection of probabilities of choosing the Early reward (as in (10)), with at least two different consumptions per date (but potentially a finite set of dates $\mathcal{T}$), and with choices on lotteries with payouts at time 0. This social scientist will estimate a discount function $\hat{D}(t)$ equal to our (normalized) as-if discount function $D(t)/D(0)$, even though the agent doesn’t actually discount utils.

The proof (in the appendix) rests on the fact that the fitted Probit model predicts exactly the probability distribution of choices made by the agent, given by equation (12).¹³

3.1 No willingness to pay for commitment

The agents in this model have a forecasting problem, not a self-control problem. Therefore they are never willing to reduce their choice set (unless they are paid to do so). This absence of a willingness to pay for commitment may explain the lack of commitment technologies in markets. In real markets, there is little commitment for commitment’s sake.¹⁴ Personal trainers and website blocking apps are frequently mentioned exceptions, but such technologies are not commonly used.

By contrast, economists have been able to elicit commitment in experiments (see Cohen, Ericson, Laibson and White, 2020 for a review). However, most of these experiments elicit only a weak taste for commitment and little or no willingness to pay for commitment (e.g., Augenblick, Niederle and Sprenger 2015; Sadoff, Samek and Sprenger 2020).

Our myopia model predicts that agents will exhibit as-if hyperbolic discounting with preference reversals *and* no willingness to pay for commitment. In this sense, our model reproduces the predictions of the standard hyperbolic discounting model with naive beliefs (see O’Donoghue and Rabin 1999a,b, 2001; Laibson 2015; Ericson 2017). However, it also generates further implications, to which we now turn.

3.2 Intelligence is Associated with Less As-if Discounting

Our model predicts that agents with less forecasting noise will exhibit less as-if discounting. Because of this mechanism, agents who are more intelligent will exhibit less as-if discounting.¹⁵

To see this formally, let H represent human capital and assume that the variance of forecasting

¹³In addition, the scientist recovers the correct utility function, up to an affine transformation.

¹⁴However, there is a great deal of ancillary commitment, like mortgage contracts, which create a forced savings system as a by-product of a stream of loan/principal repayments.

¹⁵The underlying assumption is that more intelligent agents simulate the future with less noise—for instance because they generate more simulations. If they run n simulations, the variance of the average will decrease by a factor of $1/n$, so it will be lower.

noise is declining in human capital:

$$\frac{d\sigma_{\varepsilon}^2(H)}{dH} < 0.$$

The as-if discount rate is given by

$$-\frac{D'(t)}{D(t)} = \frac{\alpha}{1 + \alpha t},$$

where

$$\alpha = \frac{\sigma_{\varepsilon}^2(H)}{\sigma_u^2}.$$

The as-if discount rate is increasing in $\sigma_{\varepsilon}^2(H)$, so as-if discounting is decreasing in human capital, H .

The available evidence supports this prediction. Measured discount rates are negatively correlated with scores on IQ tests: see Benjamin et al. (2013); Shamosh and Gray (2008); Burks et al. (2009); Dohmen et al. (2010).¹⁶ Indeed, such effects also arise across species. Tobin and Logue (1994) show that patience increases as the study population switches from pigeons, to rats, to humans. Also consistent with this prediction, Schilbach (2019) finds that reduced alcohol consumption (which increases effective IQ, H) leads to greater measured patience. Relatedly, our framework predicts that higher levels of task-specific experience will reduce discounting. Bisin and Hyndman (2020) show that procrastination declines with task experience.

3.3 Myopia Is Domain Specific

These comparative statics on cognitive function generate a wider set of predictions when forecasting ability varies across domains. For example, our framework predicts that agents with more domain-relevant experience, and hence better within-domain forecasting ability, will exhibit less discounting. Read, Frederick and Scholten (2013) report that people exhibit more patience when an intertemporal choice is posed as an investment rather than as a (seemingly novel) money-now-vs-money-later decision. Relatedly, recall our earlier discussion of the monkey experiments reported by Blanchard and Hayden (2015): when an intertemporal choice is presented as a reward-now-vs-reward-later decision, monkeys choose far more impatiently than when it is framed as a foraging problem.

Likewise, our framework predicts that older agents – who generally have more life experience and consequently better forecasting skills – will exhibit less discounting. This prediction is supported by Green et al. (1994). Relatedly, Addessi et al. (2014) show that replacing one-for-one representations of future rewards with more abstract one-for-many representations of the same future rewards leads capuchin monkeys and (human) children to exhibit more impatience. In contrast, adults, who have more experience using abstract symbols, do *not* behave more impatiently when one-for-one

¹⁶Relatedly, the ability to make complex financial decisions predicts wealth formation (Barth et al. (2020)).

representations of future rewards are replaced with one-for-many representations. The Addessi et al. (2014) experimental evidence implies that childhood impatience is due, at least in part, to children’s less developed ability to cognitively represent (abstract) future rewards. Our framework also predicts that people who experience cognitive decline (e.g., due to normal aging) will exhibit more discounting, which is supported by evidence from James et al. (2015).

Our framework predicts that agents who are *unable* to think carefully about an intertemporal tradeoff – e.g., due to a cognitive load manipulation or the effects of alcohol – will exhibit more discounting. Steele and Josephs (1990); Shiv and Fedorikhin (1999); Hinson et al. (2003); Benjamin et al. (2013) document such effects. This prediction is closely related to the theory of cognitive scarcity: see Spears (2012); Mullainathan and Shafir (2013); Schilbach et al. (2016).

Our framework predicts that agents who are encouraged to spend more time thinking about a future tradeoff will exhibit less discounting. Imas et al. (Forthcoming) robustly measure such an effect experimentally. In their experiment, some subjects decide at time 0 how to divide an effort task between time 0 and time t . Other subjects are given a preceding hour to decide how to divide the effort task between time 0 and time t . Subjects in the latter condition choose more patiently: their measured discount rate is 16 percentage points higher. The authors also find additional evidence that the additional decision time has little impact on decisions outside the domain of the task allocation and that, in the treatment that mandated a longer decision time, subjects measured to have higher information processing capacities are relatively more patient.

Our framework also predicts that rewards delivered in future periods that are cognitively well-simulated will generate less discounting. Peters and Büchel (2010) exogenously manipulate the salience of various future periods and find that higher salience from imagery of future reward periods increases the value of rewards delivered during those periods.

Our framework predicts that discounting behavior will only be weakly correlated across domains because discounting is not a domain-general preference, but rather the result of imperfect forecasting that will naturally vary across domains. Chapman (1996) and Chabris et al. (2008) document the low level of correlation in discount rates that are measured in different decision-making domains.

Finally, our framework predicts that intertemporal choices will be more patient in domains where the stakes are large and agents therefore have an incentive to create more accurate representations of the future. Gershman and Bhui (2020) develop and empirically confirm this hypothesis.

3.4 Interventions to Improve Forecasting Change Behavior

Our analysis predicts that improving forecasts may have substantial behavioral consequences, which would likely also improve welfare. For example, Liebman and Luttmer (2015) educate older workers about the increase in Social Security payments that occurs when retirement is delayed; this

intervention causes labor supply to increase by four percentage points in the year after the intervention.¹⁷ Economically large benefits have also been generated by interventions that help people more accurately visualize the future (John and Orkin (2022); Ashraf et al. (2022)). Relatedly, greater patience (e.g., healthier choice of groceries) has been elicited in the lab and the field by encouraging deliberation before intertemporal decisions (Imas et al. (Forthcoming); Brownback et al. (2021)). Understanding the limitations of people’s future forecasts may lead to targeted educational interventions that either identify specific omissions in our cognitive representations (Karlan et al. (2016); Liebman and Luttmer (2015)) or provide cognitive strategies/training that improve domain-general forecasting (e.g., Alan and Ertac (2018); John and Orkin (2022); Ashraf et al. (2022)).

For example, if first- and second-year college students think about the choice of an academic major by heavily weighting the proxy of what will make their near-term future enjoyable, they may underweight the consequences of that curricular choice for their long-run well-being. Likewise, students can likely easily forecast the near-term monetary benefits of taking a summer job but may not as easily foresee the human capital that different types of summer employment create. Similarly, a full-time worker might understand how negotiating for a raise improves short-run income, but may not realize that this raise to their current salary is worth an order of magnitude more in the long-run when the spillover effects on future wages are factored in. In cases like these, interventions may improve decision-making. Simulating complex future consequences may be a general skill that could be taught, and a habit of mind that could be developed.

4 Complementary Mechanisms that People Use to Create Cognitive Representations of the Future

Until now, we have assumed that people base their representations of future utils on noisy signals. Naturally, there are many other (related) ways that people represent the future. In section 4.1, we discuss proxy value representations of the future (e.g., using a cognitively accessible current value of a reward to estimate a more opaque future value of that reward). In Section 4.2, we discuss constructive representations of the future (e.g., putting together the building blocks of a future scenario, which necessarily implies that some elements are overlooked). Sections 4.1 and 4.2 discuss some of the psychological foundations of cognitive representation, which are complementary to the basic model that we started with in Section 3. Here, Bayesian agents combine noisy signals, proxy values, and constructive simulations to predict future events. Throughout Section 4, we

¹⁷The literature on financial education also contains many examples of low-impact interventions: e.g., see Hastings et al. (2013) and Fernandes et al. (2014). The impact of financial education interventions is actively debated, with some authors arguing for high levels of efficacy when studies with inappropriate methodology are screened out; see Kaiser et al. (2021).

discuss additional insights that emerge from these complementary microfoundations for cognitive representations of the future.

4.1 Using Current Values to Imagine Future Values: Endogenous Link Between Myopia and Projection Bias

To predict the utility of a future good, the current utility of the “same” good may be useful. To explore this, we model the dynamic utility process as:

$$u_t = \mu + v_t + w_t \quad (14)$$

where v_t has autocorrelation ρ , and w_t is i.i.d. noise. Hence, v_t captures the varying utility of the good, whereas w_t is completely transitory noise.

We assume that the agent knows u_0 (from her subjective experience or simulation) with no noise, but she does not know separately what comes from the persistent component v_0 versus the transitory component w_0 . She also receives a noisy signal, $s_t = u_t + \varepsilon_t$, about the utility at time t .¹⁸ Psychologically, we know current utility u_0 more precisely than the future utility u_t because many “situational” determinants (e.g., how thirsty we are) are known/experienced now, but hard to predict in the future.

Proposition 5 *In the non-i.i.d. case, if the agent generates a mental simulation s_t and also knows u_0 , then the agent’s as-if discount function is:*

$$D(t) = \frac{1}{1 + \frac{\sigma_{\varepsilon_t}^2}{\sigma_u^2 + \rho^t \sigma_v^2}}. \quad (15)$$

Furthermore, the agent’s point estimate of the value of the good is

$$E[u_t | s_t, u_0] = \lambda_t s_t + \pi_t u_0 \quad (16)$$

where the “projection bias” π_t is equal to

$$\pi_t = \frac{\rho^t \sigma_v^2 \sigma_{\varepsilon_t}^2}{\sigma_u^2 (\sigma_u^2 + \sigma_{\varepsilon_t}^2) - \rho^{2t} \sigma_v^4} \quad (17)$$

and λ_t is given in equation (37).

Our base case corresponds to $\rho = 0$ or $\sigma_v = 0$; in that case (15) matches (6). We now generalize

¹⁸For simplicity, we assume that the agent knows $u_0 = \mu + v_0 + w_0$ but not v_0 and w_0 separately.

our base case by assuming $\rho \geq 0$. As the persistent component, σ_v^2 , increases, the discount function $D(t)$ increases (because more is known about u_t , so there is less discounting). If $\rho \in (0, 1)$, $D(t)$ falls with t (even if we keep $\sigma_{\varepsilon_t}^2$ constant): as the horizon increases, u_0 is less informative about u_t , so the agent knows less, effectively, about u_t .

As in (3), a natural benchmark case is that the variance, $\sigma_{\varepsilon_t}^2$, increases linearly with time with a zero intercept, i.e. $\sigma_{\varepsilon_t}^2 = t\sigma^2$. Then, as before, the as-if discount function is asymptotically hyperbolic. It is everywhere hyperbolic when $\rho = 0$ or $\sigma_v = 0$.

Another instructive special case is when $\rho = 1$ and $\sigma_{\varepsilon_t} = \bar{\sigma}1_{t>0}$, i.e., when there is constant noise at any horizon. For this case, we recover a β, δ model with $\delta = 1$ and $\beta = D(t) = \frac{1}{1 + \frac{\bar{\sigma}^2}{\sigma_u^2 + \sigma_v^2}}$. The difference between the present and the future is that present utility is clearly perceived (in part because situational determinants are known), while future utilities are perceived with noise.

This model generates “projection bias” $\pi_t > 0$: given that perception of future utils is difficult, the agent uses current utility for the good as a proxy to predict future utility. Projection bias is the empirical finding that current utility (e.g., how hungry we are now) is used to project the future (e.g., how hungry we will be in the future) — see Loewenstein et al. (2003); Busse et al. (2013); Chang et al. (2018). This model generates such a phenomenon with the term π_t as in (16)-(17). In our framework, projection “bias” arises endogenously and may be an optimal Bayesian inference about future utility. However, in our framework, such projections are made with excessive weight on the present (relative to the Bayesian benchmark) when the decision-maker overestimates ρ , perhaps because the true value of ρ is very low.

As the simulation noise, $\sigma_{\varepsilon_t}^2$, increases, the weight on the present, π_t , increases (by (15)), and $D(t)$ falls (by (15)). This is because the signal about the future is less informative, so the agent should put less weight on the signal about the future and relatively more weight on the present (higher π_t).¹⁹ As the autocorrelation of utility, ρ , increases, π_t increases (as the present is more informative), and $D(t)$ increases (as there is more information about the future).

4.2 Building-Block Representation

4.2.1 Using building blocks to simulate the future

In this subsection, we introduce a complementary, related psychological model of forecasting. We now assume that the future is composed of building blocks that the economic agent tries to bring to mind to generate a forecast of the future. We assume that the future at horizon t is comprised

¹⁹Indeed, in (17), the constant in the denominator is $\sigma_u^2\sigma_u^2 - \rho^{2t}\sigma_v^4 > \sigma_u^4 - \sigma_v^4$, which is positive, as (14) implies $\sigma_u > \sigma_v$.

of N components

$$u_t = \mu + \sum_{i=1}^N b_{it} \quad (18)$$

where b_{it} are “building blocks” that jointly form (along with the additive factor μ) a complete picture of future utility u_t . We assume that the b_{it} have a data-generating process with Gaussian noise; they are i.i.d., mean 0 with $\text{var}(b_{it}) = \sigma^2$.²⁰ Hence $\text{var}(u_t) = N\sigma^2$.^{21,22} For simplicity, we assume that with enough cognitive effort, the value of every b_{it} is knowable to the decision maker. There might also be some blocks that are not knowable ex ante, no matter the cognitive investment, but we discard that, as it is not essential for what follows.

Our key assumption is that agents need to simulate the future to make forecasts. Specifically, an agent forecasting an event at horizon t sees only an endogenous number $n_t \leq N$ of the N building blocks that form a complete forecast of the future utility u_t . As a result, the as-if discounting function is as follows.

Proposition 6 *In the construction model, the discount function is:*

$$D(t) = \frac{n_t}{N}. \quad (19)$$

The proof is in Appendix A. The intuition is as follows: as the agent knows only n out of the N building blocks, the sensitivity to the true value u_t is dampened by a multiplicative factor $\frac{n_t}{N}$: if some feature moves by 1 util, the agent will perceive it only with probability $\frac{n_t}{N}$.

The “noisy signal” model of Section 2 and this “construction of the representation via partial information” model are thus mathematically isomorphic when the distribution of shocks is Gaussian. This construction model is a psychological metaphor for the rich mental representations that arise in cognition.

4.2.2 Endogenizing the richness of the representation of the future

We next endogenize the richness of the representation of the future (the number of building blocks n_t) and then derive implications. The agent has access to the following cognitive search technology: the agent cognitively simulates n_t components, and returns the corresponding estimated u_t^e .²³ We

²⁰One could imagine that there is a bias in the selection of building blocks: e.g., people tend to overlook/omit small events and, possibly, negative events (the latter of which would lead to overoptimism), but we do not pursue either of these extensions in the present paper.

²¹Note that we have assumed that the number of components, N , is the same regardless of the horizon t . One can generalize this setup by assuming that N varies with the horizon t (e.g. N_t would plausibly increase with t). Such generalizations are not going to change fundamentally the results that follow.

²²If n_t and N are large, this assumption can be weakened by appealing to the central limit theorem.

²³This is, $u_t^e = \mu + \sum_{i \in N_t} b_{it}$, where N_t is the set (with n_t elements) of blocks simulated by the agent.

illustrate these ideas in the case where reducing the imprecision $E[(u_t - u_t^e)^2] = \sigma^2(N - n_t)$ leads to a benefit (i.e., an increase in utility, excluding cognitive costs) $V(n) = Bn_t$ with $B \geq 0$.²⁴ If the cost of simulation is $C(n_t, t)$, then the agent solves:

$$\max_{n_t \leq N} V(n_t) - C(n_t, t). \quad (21)$$

We assume that the cognitive simulation cost $C(n_t, t)$ increases with the richness of the simulation (n_t) and the degree to which the simulation requires the imagination of states of nature that are far away temporally from the easily accessible present state of nature. We study the special case in which the cognitive burden is increasing in the product $n_t t$, which is a simple way to capture this idea that richness and horizon jointly contribute to simulation cost (with complementarity, i.e., a positive cross-partial derivative). We next consider a few illustrative special cases.

Some instructive special cases Suppose that the agent has $\bar{C}$ units of cognitive capacity (for instance, working memory), and that simulating n_t dimensions up to period t requires $n_t t$ steps.²⁵ The cost function is thus $C(n, t) = 0$ for $nt \leq \bar{C}$ and is infinite for $nt > \bar{C}$. Then, $n = \min\left(\frac{\bar{C}}{t}, N\right)$, and

$$D(t) = \min(D^\dagger(t), 1) \quad (22)$$

where

$$D^\dagger(t) = \frac{\bar{C}}{Nt} \quad (23)$$

We obtain a hyperbola for large enough t . If the problem comprises a greater number of dimensions (N), then the discount factor is lower. In what follows, we will also refer to $D^\dagger(t)$ as an as-if discount function. The exact discount function $D(t)$ is closely related to $D^\dagger(t)$ (see (22)), and has a maximum value of 1, which is reached when all the building blocks are simulated by the agent.

²⁴This is the case, for instance, for a consumer with total utility net of cognitive cost

$$U(q, u_t) = u_t q - \frac{\gamma}{2} q^2 - p q \quad (20)$$

when buying q quantities of a good with utility parametrized by u_t and price p . If the consumer knows an estimate $u_t^e = E[u_t | I_t]$ of the true utility u_t , she buys $q = \frac{u_t^e - p}{\gamma}$ units of the good. The utility loss compared to the utility she would get if she knew u_t is

$$E[U(q(u_t^e), u_t) - U(q(u_t), u_t)] = -\frac{1}{2\gamma} E[(u_t - u_t^e)^2] = -\frac{1}{2\gamma} \sigma^2(N - n_t).$$

Thus the benefit function is $V(n) = Bn_t$ with $B = \frac{\sigma^2}{2\gamma}$. The same phenomenon holds, as a Taylor expansion, in more general smooth convex problems, which are locally linear-quadratic.

²⁵For instance, each dimension requires a step-by-step simulation to go from one future date to the next.

Benchmark analysis We next assume that the cost and benefit of building a mental simulation of n building blocks for a good t periods in the future are given by:

$$C(n, t) = \frac{n^{1+\alpha}}{1+\alpha} t^{\alpha'}, \quad V(n, t) = B \frac{n^{1-\beta}}{1-\beta} t^{\beta'} \quad (24)$$

with α , β , and α' weakly positive and $\alpha + \beta > 0$. A natural benchmark is $\beta' = 0$, which implies that the temporal horizon is neutral in terms of material payoffs from the accuracy of the construction. However, our framework in this subsection admits cases in which $\beta' < 0$ (i.e. the future matters materially less than the present in terms of fundamental preferences), as well as the case where $\beta' > 0$ (the future matters more). We obtain the following.

Proposition 7 (As-if discount function) *When agents optimize the accuracy of their representation of the future, we obtain the as-if discount function*

$$D^\dagger(t) = \frac{k}{t^\nu}, \quad \nu = \frac{\alpha' - \beta'}{\alpha + \beta}, \quad k = \frac{B^{\frac{1}{\alpha+\beta}}}{N} \quad (25)$$

In the leading case of temporally neutral benefits ($\beta' = 0$), we obtain generalized hyperbolic discounting, where we call (following Loewenstein and Prelec (1992)) a generalized hyperbola any function of the type $1/(a + bt)^\nu$. This remains true when $\beta' \in [0, \alpha']$.²⁶ We next derive the total effort devoted to simulating a future event.

Proposition 8 (Total effort devoted to simulating a future event) *When agents optimize the richness of their representation of the future, the total simulation cost is*

$$C = K t^\kappa, \quad \kappa = \frac{\alpha'(\beta - 1) + \beta'(1 + \alpha)}{\alpha + \beta}, \quad K = \frac{B^{\frac{\alpha+1}{\alpha+\beta}}}{1 + \alpha}. \quad (26)$$

In particular, when $\kappa < 0$, agents think less about events as the event horizon increases.

We keep $\beta' = 0$ in the rest of this discussion. Proposition 8 gives the total effort, C , which varies as $C = K t^\kappa$ with $\kappa = \frac{\alpha'(\beta-1)}{\alpha+\beta}$. In the simple quadratic-loss case discussed earlier in this subsection, we have $\beta = 0$. In that case, and more generally when $\beta < 1$, then $\kappa < 0$. People spend fewer cognitive resources on events that are more distant. This is because the simulation costs are increasing in the horizon (i.e. $\alpha' > 0$), and the marginal benefit of simulating is not rapidly increasing as n falls ($\beta < 1$). Hence, decision makers spend more time evaluating opportunities in

²⁶If the future is more important than present ($\beta' > \alpha'$), we obtain anti-discounting ($\nu < 0$ in (25)). To clarify, this works through the channel of cognitive effort: because the future is more important than the present ($\beta' > \alpha'$), the agent invests more cognitive resources into simulating the future, which causes current choices to be more responsive to changes in future rewards.

the near future than opportunities in the distant future. Patterns like this—thinking less about the future than the present—have been described as indicators of irrationality, but in our setting they are the consequence of optimal allocation of cognitive effort.

This effect is reversed ($\kappa > 0$) when $\beta > 1$. It is still true that higher horizon t implies a lower endogenous simulation accuracy n , but the fall in n is mitigated by the high elasticity in the benefit function, implying that total costs of simulation rise as t increases.

Other cost functions At this time, we do not have evidence on the functional form of the cognitive cost function $C(n, t)$. We hypothesize that it is convex but anticipate that it varies across settings. With this future research in mind, we quickly review two final cases that highlight different illustrative functional forms for the cost function. For presentational simplicity, we assume a linear benefit function $V(n) = n$ in this analysis.

If $C(n_t, t)$ is independent of t , the optimal $n_t = n^*$ is independent of t , and we obtain an exact β, δ model with $\delta = 1$ and $\beta = \frac{n^*}{N}$:

$$D^\dagger(t) = \begin{cases} 1 & \text{if } t = 0 \\ \beta & \text{if } t > 0. \end{cases} \quad (27)$$

and $D(t) = D^\dagger(t)$.

If $C(n_t, t) = \frac{\kappa}{1+\alpha} n_t^{1+\alpha} e^{\gamma t}$ then we obtain an exponential as-if discounting function:

$$D^\dagger(t) = k e^{-\rho t}, \quad \rho = \frac{\gamma}{\alpha}, \quad k = \kappa^{-\frac{1}{\alpha}}. \quad (28)$$

Empirical work will be necessary to determine the form and domain-sensitivity of the cognitive cost function as well as the form and domain-sensitivity of the benefits from higher levels of forecast accuracy. In the next subsection, we discuss this research program.

4.2.3 Empirical measurement

Our analysis of imperfect forecasting and the resulting phenomenon of “as-if” discounting leads to three natural empirical questions.

First, how do people cognitively represent future events? To answer this question, social scientists will need to elicit people’s forecasts about the future and possibly also measure the cognitive mechanisms that people deploy to imagine and simulate the future. For example, Peters and Büchel (2011) and Bulley and Schacter (2020) summarize existing work along these lines.²⁷

²⁷One example of this cognitive psychology approach was motivated in part by an earlier version of this paper (Gabaix and Laibson (2017)). Hunter et al. (2018) find that experimental participants who spend more time thinking

Second, how do a typical person’s forecasts about the future compare to expert forecasts (e.g., retirement income) and compare to realized outcomes (e.g., Kahneman and Tversky (1979); Benjamin et al. (2013))? Such comparisons could be used to identify both noise and bias in forecasting. This second research program is complicated by the fact that experts are also severely limited in their ability to forecast the future, and there may be domains in which non-experts are better at predicting future utility than experts (e.g., personal preferences for future pizza). Measuring utility outcomes is also an open area of inquiry with many unresolved methodological debates.

Third, we may be able to use the insights gleaned from answering the first two questions to design interventions that improve the accuracy of people’s forecasts (as discussed in subsection 3.4).

5 Extension to a Continuous Action

Until now we have studied the case in which the agent has two mutually exclusive actions: choose an *Early* reward or a *Late* reward. We now generalize the action space to a continuum. We then provide sufficient conditions that enable us to apply our framework to a general, multi-period intertemporal choice problem. This section shows how to generalize the economic environment of the binary action case and reproduce the key results derived for that case.

5.1 Modelling How Agents Observe with Noise a Whole Utility Function

Suppose that an action a leads to a true payoff $u(a)$. However, the agent observes this noisily: we suppose that the agent observes the “noised-up” version of the utility function:

$$\mathbf{s} = (s(a))_{a \in \mathcal{A}} \tag{29}$$

of the whole function $u = (u(a))_{a \in \mathcal{A}}$, where $\mathcal{A} = [\underline{a}, \bar{a}]$ is the support of the action, which is assumed to contain 0 (this is just a normalization). This noised-up version is assumed to take the form:

$$s(a) = u(a) + \sigma_{\varepsilon_t} W(a) + \chi \sigma_{\varepsilon_t} \eta_0 \tag{30}$$

for all $a \in \mathcal{A}$. There is a continuous noise $W(a)$, modeled as standard Brownian motion with $W(0) = 0$, except that W is “two-sided”, i.e. runs to the left and right of 0.^{28,29} The noise is

about their decision in a money-earlier-vs-later task are also more likely to make (forward-looking) multi-stage plans in a sequential reinforcement learning task.

²⁸Formally, $W(x)_{x \geq 0}$ and $W(-x)_{x \geq 0}$ are independent Brownian motions.

²⁹See Callander and Hummel (2014) for a model using inference on Brownian paths, though with a signal structure different from ours.

modeled as proportional to σ_{ε_t} when the utility is seen from a distance t . For instance, the linear case is $\sigma_{\varepsilon_t} = \sigma_\varepsilon \sqrt{t}$. The term $\chi \sigma_{\varepsilon_t} \eta_0$ ensures that the value at $a = 0$ is also perceived with noise (χ is a parameter, and $\eta_0 \sim \mathcal{N}(0, 1)$).

Given this perceived curve $\mathbf{s}$, what is the agent’s posterior about $u(a)$? We will see that under the “right” assumptions (to be specified soon), we simply have

$$\mathbb{E}[u(a) \mid \mathbf{s}] = D(t) s(a)$$

with the same $D(t)$ as in the binary case. This means that the representative agent just dampens the true function.

5.2 Assumptions for Our Result

Here, we detail the assumptions we use for the results. The reader may wish to skip to the result itself in the next subsection, 5.3.

Assumption 1 (Wiener decomposition) *We suppose that the utility is given by $u(a) = u(0) + \sigma_u v(a)$, where $v(a) := \frac{u(a) - u(0)}{\sigma_u}$ is drawn from the Wiener measure, and $u(0)$ is drawn as $u(0) \sim N(0, \chi^2 \sigma_u^2)$ independent of v . We call*

$$D(t) = \frac{1}{1 + \frac{\sigma_{\varepsilon_t}^2}{\sigma_u^2}} \tag{31}$$

where σ_{ε_t} is as in (30).

Let us state this assumption in more user-friendly language. The value of $u(0)$ is also seen as random—and we parametrize its variance randomness by χ^2 . The rest of the function u , outside the intercept, is also random. To specify this, we set $v(a) := \frac{u(a) - u(0)}{\sigma_u}$, which is u normalized to have 0 intercept and standardized size (so that $\mathbb{E}[v(1)^2] = 1$). We view v as a “random function” drawn from a distribution. For simplicity, we consider that it’s drawn from the simplest distribution of random functions – the so-called Wiener measure (Brownian motions are typical instances of such functions).³⁰ Basically, the assumption is that the components of $du(a)$ are drawn as i.i.d. normal increments, like a Brownian motion, with square width $\sigma_u^2 da$. Note that this refers to the distribution assumed by the agent when he performs his Bayesian inference, not necessarily to the true distribution.

³⁰We could imagine a number of variants, e.g. u'' would be drawn from this distribution; or, to keep u concave, we could have $\ln(-u'')$ drawn from this distribution. This becomes quickly more mathematically involved, so we leave this to a separate investigation and focus on what we view as the simplest case.

Section B of the appendix proposes a variant, Assumption 2, with polynomial utility, that uses more elementary mathematics, at the cost of heavier notations and proofs.

5.3 Perceived Utility Function Given the True Utility

We can now derive the utility perceived by the agent, given that she sees the whole noised-up function $\mathbf{s}$ (equation (30)).

Proposition 9 (Perceived utility for a continuous utility function) *Make Assumption 1 or 2. Then, the perceived utility is proportional to the signal:*

$$\mathbb{E}[u(a) \mid \mathbf{s}] = D(t) s(a)$$

where $D(t) = \frac{1}{1 + \frac{\sigma_{\varepsilon_t}^2}{\sigma_u^2}}$. As a result, the average perceived utility $\bar{u}(a)$, defined as:

$$\bar{u}(a) := \mathbb{E}[\mathbb{E}[u(a) \mid \mathbf{s}] \mid \mathbf{u}]$$

satisfies:

$$\bar{u}(a) = D(t) u(a) \tag{32}$$

This means that the average perceived utility is $D(t) u(a)$ rather than plainly $u(a)$, exactly like in the simple two-action (consume or don't consume) case.

5.4 The Representative Agent Perspective

To cleanly study the dynamic problem, we define a “representative agent” by assuming the following (in addition to the assumptions of Proposition 9).³¹

- A1. The agent treats the noise at all simulation horizons as uncorrelated.
- A2. The agent has Gaussian priors with 0 mean (and no correlation between $u_t, u_{t+\tau}$).
- A3. The agent acts as if she won't learn new simulation information in the future.³²

The notion of “average behavior” is potentially messy with non-linear utilities. Hence, we find it useful to define the following form of a “representative agent” version of the model. We study the

³¹There are many alternative ways to generate variances that are linear in the forecasting horizon, including new signals that contain all of the information of the old signals.

³²This is the assumption of the “anticipated utility” framework of Kreps (1998) used also by Cogley and Sargent (2008).

equilibrium path in which all simulation noise happens to be realized as zero (but the agent doesn't know this). In our illustrative example, this corresponds to $\varepsilon_t = 0$. For instance, we had $s_t = u_t + \varepsilon_t$ and $\mathbb{E}[u_t \mid s_t] = D(t)s_t$. The representative agent draws noise $\varepsilon_t = 0$, so for the representative agent, $\mathbb{E}[u_t \mid s_t] = D(t)u_t$.

Proposition 10 (Dynamic choices of the representative agent) *Assume that the agent has dynamically consistent preferences*

$$\sum_{t=0}^T u(a_t).$$

Then A1-A3 imply that at each time period $t \in \{0, \dots, T\}$ the representative agent acts as if she is trying to maximize

$$\sum_{\tau=0}^{T-t} D(\tau)u(a_{t+\tau})$$

where

$$D(\tau) = \frac{1}{1 + \frac{\sigma_{\varepsilon_\tau}^2}{\sigma_u^2}}.$$

Corollary 1 *Assume that simulation variance is linear in the horizon of the simulation: $\sigma_{\varepsilon_\tau}^2 = \tau\sigma_\varepsilon^2$. Then, at each time period $t \in \{0, \dots, T\}$ the representative agent acts as if she is trying to maximize*

$$\sum_{\tau=0}^{T-t} D(\tau)u(a_{t+\tau}),$$

where

$$D(\tau) = \frac{1}{1 + \alpha\tau}, \quad \alpha = \frac{\sigma_\varepsilon^2}{\sigma_u^2}.$$

Proposition 10 shows that our basic results extend to arbitrary utility functions with continuous actions.

6 Literatures on Related Mechanisms

We now review other lines of research that are related to this paper that we have not yet discussed and on which this paper builds. We review three literatures: models of myopia, Bayesian foundations of imperfect and costly cognition, and risk-based models of as-if discounting (including risk-based models with probability distortions).

6.1 Myopia

Political economists, psychologists, and other social scientists have long posited that impatient behavior is due in part to imperfect foresight. These ideas were informally described by political economists, including von Böhm-Bawerk (1890), and economists, including Pigou (1920), both of whom are quoted in our introductory section.

These informal explanations have been joined by formal, mathematical definitions, models, and analyses that incorporate various formulations of myopia. For example, Brown and Lewis (1981) provide an axiomatic definition of myopia. Feldstein (1985) evaluates the optimality of social security under the assumptions of myopia and partial myopia (modeled as a low discount factor in a two-period decision problem). Jéhiel (1995) studies two-player games in which players have limited forecasting horizons. Spears (2012) generates a forecasting horizon that is endogenous because forward-looking calculations are costly. Gabaix et al. (2006) report experimental evidence that supports a model in which agents choose an endogenous forecasting horizon at which the cognitive cost and estimated utility benefit of marginally increasing the forecasting horizon are equalized. This optimal forecasting framework generates a complex option value problem with respect to information acquisition (see also Fudenberg et al. (2018)).³³

In the current paper, we assume that the agent has noisy signals about the future, which engenders Bayesian forecasts that have “myopic” properties: i.e., declining sensitivity to future events. When the noise is linear in the forecasting horizon, the as-if discounting takes a simple hyperbolic form. Accordingly, our paper introduces a tractable microfoundation for myopia.

6.2 Models of Limited Attention and Cognitive Constraints

The current paper assumes that agents are Bayesian, which adopts the approach of early decision-theory pioneers like Raiffa and Schlaifer (1961). There is a growing body of literature (in economics, cognitive psychology, and neuroscience) that studies the effects of noisy perception and Bayesian inference and uses this combination to explain seemingly suboptimal behaviors. One of the earliest examples is the work of Commons et al. (1982) and Commons et al. (1991) who use this approach to generate a theory of hyperbolic memory recall. In their framework, the noisy signals are memories, whereas the noisy signals in our model are simulations of the future. The literature on attention allocation assumes that agents have limited information, which is mathematically equivalent to the assumption that agents have noisy signals about the state of the world. Verrecchia (1982);

³³Neuroscientists have long hypothesized that perceptions of consequences are neurally represented as noisy unbiased signals of those consequences. For example, this is the foundational assumption of the large literature on the drift-diffusion model (e.g., Shadlen and Shohamy (2016)). This framework includes a noise term analogous to our noise term. By implication, more noise in the drift-diffusion model would imply more discounting in our model.

Geanakoplos and Milgrom (1991); Gabaix and Laibson (2002); Sims (2003); Kamenica (2008); Woodford (2009); Gabaix (2014); Schwartzstein (2014); Hanna et al. (2014); Allcott and Taubinsky (2015); Steiner et al. (2017); Natenzon (2019); Taubinsky and Rees-Jones (2018); Rees-Jones and Taubinsky (2020); Angeletos and Lian (2018); Gabaix (2019); Kőszegi and Matějka (2020) study agents who allocate their limited attentional bandwidth to the activities that they believe are the most valuable.³⁴ Steiner and Stewart (2016); Khaw et al. (2021) study an environment in which agents react to the noise in their probability perceptions by (optimally) distorting their perceived probabilities in a way that mimics the probability mapping in prospect theory (Kahneman and Tversky, 1979).

Our paper adopts the approach that unifies the work above: noisy signals plus Bayesian inference jointly produce as-if behavior that appears to be imperfectly rational. Specifically, in our case, this combination generates as-if hyperbolic discounting.

6.3 Risk-Based Models of As-if Discounting

It has long been recognized that time preferences engender the same kind of behavior that is associated with risk or mortality (e.g., Yaari (1965)). For example, if promised future rewards may be permanently withdrawn or lost at a constant hazard rate, ρ , then a perfectly patient decision-maker should be indifferent between 1 util at time zero and $\exp(\rho\tau)$ utils at time τ . In this example, risk induces a perfectly patient agent to appear to be discounting the future with exponential discount rate ρ .

This type of risk-based discounting can also produce hyperboloid discount functions under specific assumptions about a non-constant hazard rate (see Sozou (1998); Azfar (1999); Weitzman (2001); Dasgupta and Maskin (2005); Fernández-Villaverde and Mukherji (2006); Halevy (2014a,b, 2015)). For instance, Azfar, Sozou, and Weitzman all assume that the hazard rate that governs the withdrawal of rewards is itself drawn from a distribution and has a value that can only be inferred from the observed data. This assumption produces preferences that are characterized by a declining discount rate as the horizon increases—the more time that passes without a withdrawal, the more likely that one of the low hazard rates is the hazard rate that was drawn from the distribution at the start of time, implying a lower effective discount rate at longer horizons. Risk can also produce hyperboloid discount functions because of probability transformations that are characterized by a certainty effect, whereby a certain present reward is discretely more valuable than an even slightly uncertain delayed reward (see the non-expected utility frameworks in Prelec and Loewenstein (1991); Quiggin and Horowitz (1995); Weber and Chapman (2005); Halevy (2008); Epper et

³⁴Another strand of the literature uses non-Bayesian rules to govern attention and salience (e.g. Bordalo et al. (2012, 2013)), though it might be given some quasi-Bayesian interpretation.

al. (2011); Baucells and Heukamp (2012); Andreoni and Sprenger (2012); Epper and Fehr-Duda (2018); Chakraborty (2021)).

Our model works off a related but different risk mechanism than those listed above. The uncertainty in our model is due to noise that is generated by the forecaster herself. For example, our mechanism predicts that an expert would exhibit little as-if discounting in her domain of expertise (she forecasts the future with little or no noise), while a non-expert would exhibit substantial as-if discounting in the *same* domain (she forecasts the future with relatively more noise than the expert). Likewise, our framework predicts that cognitive load should increase as-if discounting because it reduces an agent’s ability to forecast accurately. Accordingly, our noise-based discounting mechanism is not propagated by external risk (like mortality or the likelihood of default), but rather by noise associated with the limited forecasting ability of the decision-maker.

Finally, our framework is consistent with Bayesian decision-making and expected utility theory. Accordingly, our agent will not be dynamically inconsistent and will not pay for commitment. In our framework, preference reversals reflect classical information acquisition, not weakness of will.

Our key assumption is that the agent has (unbiased) noise in her signals about the future. This noise leads our agent to optimally down-weight her simulations of the future and therefore place more weight on her priors. Consequently, she ends up being (rationally) imperfectly responsive to future contingencies and therefore behaves as if she discounts the future. As her expertise and experience improve (over her lifetime, or as she gains domain-specific knowledge), she shifts her behavior and acts as-if she has become more patient.

6.4 True Discounting vs. Extrinsic Risk vs. Myopia

We have now summarized three mechanisms that induce declining sensitivity to delayed utility flows: (i) true discounting arising from deep time preferences (e.g., exponential discounting, present bias, or some other time preference function); (ii) extrinsic risk (e.g., mortality, default, or some other source of risk); and (iii) *myopia* arising from forecasting noise (the focus of the current paper).

The reader may wonder how one can pull these mechanisms apart empirically. Extrinsic risk is the easiest to distinguish from the other two. Extrinsic risk can be measured directly (e.g., by measuring sources of extrinsic risk), but it can also be measured indirectly by studying agents’ beliefs. For example, a researcher could elicit an agent’s subjective probability that she will fail to receive a promised payment at a future date. Subjects might report that they trust an immediate payment more than a delayed payment (e.g., payable in a year’s time). Failing to trust a delayed payment is an example of (perceived) extrinsic risk. Extrinsic risk is associated with “flat” learning dynamics in the sense that experience/expertise may either reduce or increase perceptions of extrinsic risk. For example, a merchant may discover through experience that a customer is reliable (unreliable),

thereby leading the merchant to increase (decrease) credit to the customer. Likewise, an experimental subject may learn to trust (distrust) an experimenter, leading the subject to increase (decrease) her willingness to choose larger, later rewards over smaller, immediate rewards.

True time discounting is easy to conflate empirically with myopia. For example, we have shown that (true) hyperbolic discounting with naivet   is observationally equivalent to myopia with linear simulation noise (see Propositions 2 and 4).

Despite this similarity, true time discounting and myopia induce very different learning dynamics. With true time discounting, learning/experience/expertise induce *no* change in the underlying time preferences. In the presence of myopia, learning/experience/expertise induce *less* as-if discounting because the simulation noise falls and/or because the priors become more refined and pull away from zero. For example, consider a household/subject who is thinking about some future opportunity. The more the household thinks about the future decision, the more responsive the household will be to future tradeoffs and the less as-if discounting the household will exhibit (e.g., Imas et al. (Forthcoming)).

Learning can also be used to identify the existence of true time discounting that is dynamically inconsistent (and therefore induces a self-control problem, such as present-bias). If time preferences are dynamically inconsistent, learning will generate a preference for commitment (e.g., a 50-year-old who plays too much computer chess may finally realize that he needs to delete the app from his iPad to stop himself from playing too much). With myopia, by contrast, learning does not generate a preference for commitment. The importance of this distinction may be played out in policy design, where, for example, both the preference for commitment and the true time discounting rate impact the effectiveness of tax payment delays in improving the welfare of lower income households (Lockwood (2020)).

In summary, it is possible to empirically distinguish between true time preferences and myopia by studying learning dynamics. With true time preferences, learning generates no change in the time preferences, and, if the time preferences are dynamically inconsistent, learning engenders a taste for commitment. With myopia, learning generates less (as-if) discounting and no taste for commitment.

7 Conclusion

In our baseline analysis, we assume that *perfectly patient* agents estimate the value of future events by generating noisy, unbiased simulations of those events. Our agents combine these noisy signals with their priors, thereby forming posterior utility expectations. We show that these expectations exhibit a property that we call *as-if discounting*. Specifically, the agent makes choices as if she

were maximizing a stream of known utils weighted by an as-if discount function, $D(t)$. This as-if discount function adjusts for the fact that future utils are not actually known by the agent and must be estimated with noisy signals and priors. This estimation shades the estimated utils toward the mean of the prior distribution, creating behavior that largely mimics the effect of classical time preferences.

When the simulation noise has a variance that is linear in the event’s horizon, the as-if discount function is hyperbolic:

$$D(t) = \frac{1}{1 + \alpha t},$$

where α is the ratio of the variance of (per-period) simulation noise to the variance of events in the agent’s prior distribution.

We also analyze two additional extensions, which work out some additional related mechanisms that underlie imperfect forecasting. In one extension, perfectly patient agents use proxy value representations of the future: e.g., using a relatively cognitively accessible current value of a reward to estimate a more opaque long-run/future value of a reward. In the other extension, agents form constructive representations of the future by mentally aggregating the key building blocks of a future scenario and cognitively overlooking some of the myriad elements that characterize that future scenario. Our baseline model (noise) and our two extensions (proxy valuation and constructive forecasting) are mutually compatible. We believe that decision-makers simultaneously deploy and integrate these cognitive strategies. We leave such modeling to future work.

Our models generate numerous predictions that match existing empirical evidence. Our agents exhibit systematic preference reversals. Our agents have no intrinsic taste for commitment because they suffer from an imperfect forecasting problem, not a self-control problem. Our agents will exhibit comparative statics with respect to cognitive function: people who are more skilled/experienced at forecasting will exhibit less discounting.

Our framework predicts many domain-specific discounting effects. Agents with more domain-relevant experience will exhibit less discounting. Older agents (who have more experience hence are better at forecasting) will exhibit less discounting (except for those with cognitive decline, who will exhibit more discounting). Agents who are encouraged to spend more time thinking about a future tradeoff will exhibit less discounting. Finally, agents who are unable to think carefully about an intertemporal tradeoff—e.g., due to a cognitive load manipulation—will exhibit more discounting.

Our framework predicts that discounting is a highly variable and plastic phenomenon that arises from imperfect forecasting of future rewards or costs. Our model provides a complementary alternative to the classical assumption that discounting arises from a deep preference for known rewards to be moved earlier in time.

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A Appendix: Omitted Proofs

Proof of Proposition 1 This proof is very elementary, but for completeness we provide its calculations. We normalize $\mu = 0$ without loss of generality (for instance, by considering $u'_t = u_t - \mu$ and $s'_t = s_t - \mu$). It is well-known that $u_t \mid s_t$ is Gaussian distributed, and can be represented:

$$u_t = \lambda s_t + \eta_t \tag{33}$$

for some λ , and some Gaussian variable η_t independent of s_t , so that $\mathbb{E}[s_t \eta_t] = 0$. Multiplying (33) by s_t on both sides and taking the expectations gives: $\mathbb{E}[u_t s_t] = \lambda \mathbb{E}[s_t^2]$, i.e.

$$\begin{aligned} \lambda &= \frac{\mathbb{E}[u_t s_t]}{\mathbb{E}[s_t^2]} = \frac{\mathbb{E}[u_t (u_t + \varepsilon_t)]}{\mathbb{E}[(u_t + \varepsilon_t)^2]} = \frac{\mathbb{E}[u_t^2]}{\mathbb{E}[u_t^2 + \varepsilon_t^2]} \text{ as } \mathbb{E}[u_t \varepsilon_t] = 0 \\ &= \frac{\sigma_u^2}{\sigma_u^2 + \sigma_{\varepsilon_t}^2} = \frac{1}{1 + \frac{\sigma_{\varepsilon_t}^2}{\sigma_u^2}} = D(t). \end{aligned}$$

Next, taking the variance of both sides of (33), we have

$$\sigma_u^2 = \lambda^2 \sigma_s^2 + \text{var}(\eta_t)$$

as $\text{cov}(s_t, \eta_t) = 0$ and with $\sigma_s^2 = \sigma_u^2 + \sigma_{\varepsilon_t}^2$. So, using $\lambda \sigma_s^2 = \sigma_u^2$,

$$\text{var}(\eta_t) = \sigma_u^2 - \lambda^2 \sigma_s^2 = \sigma_u^2 - \lambda \sigma_u^2 = (1 - \lambda) \sigma_u^2.$$

Hence, $u_t \mid s_t \sim \mathcal{N}(\lambda s_t, (1 - \lambda) \sigma_u^2)$, as announced.

Proof of Proposition 3 By homogeneity, we can normalize $\sigma_u = 1$ without loss of generality. As $D(t) = \frac{1}{1 + \sigma_{\varepsilon_t}^2}$, its time-derivative writes:

$$D'(t) = -D(t)^2 \frac{d\sigma_{\varepsilon_t}^2}{dt},$$

so that $\rho(t) = -D'(t)/D(t) = D(t) \frac{d\sigma_{\varepsilon_t}^2}{dt}$. In turn,

$$\rho'(t) = D(t) \frac{d^2 \sigma_{\varepsilon_t}^2}{dt^2} + D'(t) \frac{d\sigma_{\varepsilon_t}^2}{dt} = D(t) \frac{d^2 \sigma_{\varepsilon_t}^2}{dt^2} - D(t)^2 \left(\frac{d\sigma_{\varepsilon_t}^2}{dt} \right)^2 = D(t) \left[\frac{d^2 \sigma_{\varepsilon_t}^2}{dt^2} - D(t) \left(\frac{d\sigma_{\varepsilon_t}^2}{dt} \right)^2 \right].$$

Hence agents exhibit strongly decreasing impatience, i.e. $\rho'(t) < 0$, if and only if

$$\frac{d^2 \sigma_{\varepsilon_t}^2}{dt^2} - \frac{\left(\frac{d\sigma_{\varepsilon_t}^2}{dt} \right)^2}{1 + \sigma_{\varepsilon_t}^2} < 0,$$

which is Proposition 3 with $\sigma_u = 1$.

Proof of Proposition 4 In this proof, we normalize μ to 0 (again, this is innocuous, replacing $u(c)$ by $u(c) - \mu$ as necessary). Because the scientist has access to the lotteries on the goods at time 0, he can recover the agent's cardinal utility up to affine transformation, i.e. he can fit

$\hat{u}(c) = K(u(c) + b)$ for all c , with constants K and b to be determined. Then, moving to choices for more general dates, the scientist needs to fit the probability of choosing *Early* for all dates $t, t + \tau$, which was derived in (10), i.e. he needs to make sure that his fitted values (all denoted with hats) satisfy:

$$\Phi\left(\frac{1}{\widehat{\Sigma}_{t,t+\tau}}\left[\widehat{D}(t)\widehat{u}(c_t) - \widehat{D}(t+\tau)\widehat{u}(c_{t+\tau})\right]\right) = \Phi\left(\frac{1}{\Sigma_{t,t+\tau}}[D(t)u(c_t) - D(t+\tau)u(c_{t+\tau})]\right)$$

Both the left- and right-hand sides of this equation represent the probability of choosing good c_t at t over good $c_{t+\tau}$ at $t + \tau$. On the left, this probability is expressed in the scientist's model (with $\widehat{\Sigma}_{t,t+\tau}$ the standard deviation of the Probit noise), and on the right, this probability is represented by the true data generating process (see (10)). In other terms, the scientist's fitted values must insure that for all consumptions and dates in his dataset:

$$\frac{1}{\widehat{\Sigma}_{t,t+\tau}}\left[\widehat{D}(t)\widehat{u}(c_t) - \widehat{D}(t+\tau)\widehat{u}(c_{t+\tau})\right] = \frac{1}{\Sigma_{t,t+\tau}}[D(t)u(c_t) - D(t+\tau)u(c_{t+\tau})]. \quad (34)$$

It is clear that a possible way to make (34) hold exactly is to have: $\widehat{D}(t) = \frac{D(t)}{D(0)}$, $\widehat{u}(c) = Ku(c)$ and $\widehat{\Sigma}_{t,t+\tau} = \frac{K\Sigma_{t,t+\tau}}{D(0)}$. Then, the two sides of (34) are equal.

This almost proves the proposition, but not quite: we also have to verify that this is the only possible fit (up to the usual arbitrary scaling factor $K > 0$). To finish the last step and prove uniqueness, take (34) for two different values of c_t , keeping the other terms constant, and subtract them. This implies $\frac{K\widehat{D}(t)}{\widehat{\Sigma}_{t,t+\tau}} = \frac{D(t)}{\Sigma_{t,t+\tau}}$, i.e. $\frac{K\Sigma_{t,t+\tau}}{\widehat{\Sigma}_{t,t+\tau}} = \frac{D(t)}{\widehat{D}(t)}$. The same reasoning applied to two different values of $c_{t+\tau}$ gives $\frac{K\Sigma_{t,t+\tau}}{\widehat{\Sigma}_{t,t+\tau}} = \frac{D(t+\tau)}{\widehat{D}(t+\tau)}$, i.e.

$$\frac{K\Sigma_{t,t+\tau}}{\widehat{\Sigma}_{t,t+\tau}} = \frac{D(t)}{\widehat{D}(t)} = \frac{D(t+\tau)}{\widehat{D}(t+\tau)}. \quad (35)$$

Using (35) for $t = 0$ gives: $D(0) = \frac{D(\tau)}{\widehat{D}(\tau)}$ (indeed, recall that we normalized $\widehat{D}(0) = 1$), i.e., $\widehat{D}(\tau) = \frac{D(\tau)}{D(0)}$. This holds for all dates τ . Then (35) implies that for all dates $t, t + \tau$, we have $\frac{K\Sigma_{t,t+\tau}}{\widehat{\Sigma}_{t,t+\tau}} = D(0)$. Finally, (34) then implies that $b = 0$. This concludes the proof of uniqueness, up to the usual multiplicative factor $K > 0$.

Proof of Proposition 5 We normalize $\mu = 0$. Because Gaussian updating is linear in means, we have

$$E[u_t | s_t, u_0] = \lambda_t s_t + \pi_t u_0$$

for some coefficients λ_t and π_t that we next calculate. Call $X_t := (s_t, u_0)$. Then

$$\text{var}(X) = E[XX'] = \begin{pmatrix} \sigma_u^2 + \sigma_{\varepsilon_t}^2 & \rho^t \sigma_v^2 \\ \rho^t \sigma_v^2 & \sigma_u^2 \end{pmatrix}$$

so that, with $\Delta = \sigma_u^2 (\sigma_u^2 + \sigma_{\varepsilon_t}^2) - \rho^{2t} \sigma_v^4$,

$$\text{var}(X)^{-1} = \frac{1}{\Delta} \begin{pmatrix} \sigma_u^2 & -\rho^t \sigma_v^2 \\ -\rho^t \sigma_v^2 & \sigma_u^2 + \sigma_{\varepsilon_t}^2 \end{pmatrix}.$$

Also,

$$E[Xu_t] = \begin{pmatrix} \sigma_u^2 \\ \rho^t \sigma_v^2 \end{pmatrix} \quad (36)$$

so the regression coefficients are

$$\begin{pmatrix} \lambda_t \\ \pi_t \end{pmatrix} = \text{var}(X)^{-1} E[Xu_t] = \frac{1}{\Delta} \begin{pmatrix} \sigma_u^2 & -\rho^t \sigma_v^2 \\ -\rho^t \sigma_v^2 & \sigma_u^2 + \sigma_{\varepsilon_t}^2 \end{pmatrix} \begin{pmatrix} \sigma_u^2 \\ \rho^t \sigma_v^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_t \\ \pi_t \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \sigma_u^4 - \rho^{2t} \sigma_v^4 \\ \rho^t \sigma_v^2 \sigma_{\varepsilon_t}^2 \end{pmatrix} \quad (37)$$

Furthermore, the agent takes the early good iff $u_0 \geq E[u_t | s_t, u_0]$, so with probability

$$P_t = P(u_0 \geq E[u_t | s_t, u_0]) = P(u_0 \geq \lambda_t s_t + \pi_t u_0) = P\left(u_0 \geq \frac{\lambda_t}{1 - \pi_t} (u_t + \varepsilon_t)\right)$$

so the as-if discount factor is $D(t) = \frac{\lambda_t}{1 - \pi_t}$. This gives:

$$D(t) = \frac{\lambda_t}{1 - \pi_t} = \frac{\lambda_t \Delta}{\Delta - \pi_t \Delta} = \frac{\sigma_u^4 - \rho^{2t} \sigma_v^4}{\sigma_u^2 (\sigma_u^2 + \sigma_{\varepsilon_t}^2) - \rho^{2t} \sigma_v^4 - \rho^t \sigma_v^2 \sigma_{\varepsilon_t}^2} \quad (38)$$

$$= \frac{1}{1 + \sigma_{\varepsilon_t}^2 \frac{\sigma_u^2 - \rho^t \sigma_v^2}{\sigma_u^4 - \rho^{2t} \sigma_v^4}} = \frac{1}{1 + \frac{\sigma_{\varepsilon_t}^2}{\sigma_u^2 + \rho^t \sigma_v^2}} \quad (39)$$

Proof of Proposition 6 The agent has access to the information $I_0 = (b_{it})_{i=1 \dots n_t}$. So, her estimate of u_t is (we again normalize $\mu = 0$)

$$u_t^e = E[u_t | I_0] = \sum_{i=1}^{n_t} b_{i,t} \quad (40)$$

As variables are jointly Gaussian, we can express u_t^e as

$$u_t^e = \Lambda_t u_t + e_t \quad (41)$$

where $\Lambda_t = \frac{\text{cov}(u_t, u_t^e)}{\text{var}(u_t)} = \frac{n_t}{N_t}$ is the regression coefficient and e_t is a residual. So, the signal can also be transformed as

$$s_t = \frac{u_t^e}{\Lambda_t} = u_t + \varepsilon_t, \quad (42)$$

with $\varepsilon_t := \frac{e_t}{\Lambda_t}$. This puts us in the basic case of Section 2.

Proof of Propositions 7 and 8 We solve $\max_n V(n, t) - C(n, t)$, whose first order condition $n^\alpha t^{\alpha'} = B n^{-\beta} t^{\beta'}$. So $n = B^{\frac{1}{\alpha+\beta}} t^{-\nu}$ with $\nu = \frac{\alpha' - \beta'}{\alpha + \beta}$. Finally, (19) gives $D(t) = \frac{n_t}{N_t}$.

The cost satisfies

$$C = \frac{n^{1+\alpha}}{1+\alpha} t^{\alpha'} = \frac{1}{1+\alpha} \left(B^{\frac{1}{\alpha+\beta}} t^{-\nu} \right)^{1+\alpha} t^{\alpha'} = \frac{B^{\frac{\alpha+1}{\alpha+\beta}}}{1+\alpha} t^\kappa$$

with

$$\kappa = -(1+\alpha)\nu + \alpha' = -(1+\alpha) \frac{\alpha' - \beta'}{\alpha + \beta} + \alpha' = \frac{\alpha'(\beta - 1) + \beta'(1 + \alpha)}{\alpha + \beta}.$$

Proof of Proposition 9 It is a corollary of Proposition 11 (for Assumption 1) and Proposition 12 (for Assumption 2) below.

Proof of Proposition 10 Given our assumptions, the agent at time t will want to maximize

$$\max_{(a_{t+\tau})_{\tau \geq 0}} \mathbb{E} \left[\sum_{\tau=0}^{T-t} u(a_{t+\tau}) \mid \mathbf{s} \right] = \max_{(a_{t+\tau})_{\tau \geq 0}} \sum_{\tau=0}^{T-t} D(\tau) s_{t+\tau}(a_{t+\tau})$$

where $\mathbf{s} = (s_t(y), \dots, s_{t+\tau}(y))_{y \in \mathcal{A}}$. Assumption A1-A3 allows us to remove expected values. For our representative agent, we have $s_{t+\tau}(a) = u_{t+\tau}(a)$. Hence, this representative agent maximizes at time t :

$$\max_{(a_{t+\tau})_{\tau \geq 0}} \sum_{\tau=0}^{T-t} D(\tau) u(a_{t+\tau}).$$

Online Appendix for “Myopia and Discounting”

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May 19, 2022

This online appendix gives complements to the continuous actions case of Section 5.

B Complements to the Continuous Actions Case

To simplify the notations, we set $\sigma = \sigma_{\varepsilon_t}$.

B.1 Result for the Wiener case

Proposition 11 (Bayesian updating with functions) *Under Assumption 1, we have*

$$\mathbb{E}[u(a) \mid \mathbf{s}] = \lambda s(a)$$

with $\lambda = \frac{1}{1+\sigma_{\varepsilon_t}^2/\sigma_u^2}$. This means that we can do Bayesian updating on this space of functions.

Proof. Take the increments:

$$ds(a) = du(a) + \sigma dW(a)$$

The key observation is that the $ds(a)$ ’s are all Gaussians innovations, independent of the value of the functions at other points $y \neq a$. So, by the formulation for Gaussian updating we used before:

$$\mathbb{E}[du(a) \mid ds(a)] = \lambda ds(a)$$

with $\lambda = \frac{1}{1+\sigma_{\varepsilon_t}^2/\sigma_u^2}$. Next, because the $du(a)$ and $dW(a)$ are independent,

$$\mathbb{E}[du(a) \mid \mathbf{s}] = \mathbb{E}[du(a) \mid ds(a)] = \lambda ds(a). \quad (43)$$

Next, the behavior at 0 needs a special treatment. Because $s(0) = u(0) + \chi\sigma\eta_0$, $\mathbb{E}[u(0) \mid s(0)] = \lambda_0 s(0)$, with $\lambda_0 = \frac{1}{1+\frac{\text{var}(\chi\sigma\eta_0)}{\text{var}(u_0)}} = \frac{1}{1+\frac{\chi^2\sigma^2}{\chi^2\sigma_u^2}} = \lambda$. Then, $\mathbb{E}[u(0) \mid s(0)] = \lambda s(0)$, and by independence:

$$\mathbb{E}[u(0) \mid \mathbf{s}] = \lambda s(0). \quad (44)$$

Hence, integrating from 0 to a , we get

$$\begin{aligned}\mathbb{E}[u(a) \mid \mathbf{s}] &= \mathbb{E}\left[u(0) + \int_{y=0}^a du(y) \mid \mathbf{s}\right] = \mathbb{E}[u(0) \mid \mathbf{s}] + \int_{y=0}^a \mathbb{E}[du(y) \mid \mathbf{s}] \\ &= \lambda s(0) + \int_0^a \lambda ds(y) \\ &= \lambda s(a).\end{aligned}$$

B.2 Polynomial utility

Here we provide assumptions that are a little more elementary, but apply only when the utility function $u(a)$ is a polynomial in a . For instance, we want to capture that $u(a) = b_0 + b_1 a + b_2 a^2$ with unknown coefficients b_i , that the agent wants to learn from noisy signals.

Assumptions for the polynomial utility case We normalize the action space to be $[\underline{a}, \bar{a}] = [-1, 1]$. We shall use the Legendre polynomials $P_i(a)$ as a basis, as they are more convenient than the plain monomials a^i . We have for instance:³⁵

$$P_0(a) = 1, \quad P_1(a) = a, \quad P_2(a) = \frac{1}{2}(3a^2 - 1), \quad P_3(a) = \frac{1}{2}(5a^3 - 3a).$$

Using the inner product on $L^2([-1, 1])$:

$$\langle f \mid g \rangle := \int_{-1}^1 f(a) g(a) da, \tag{45}$$

we have the standard orthogonality result: $\langle P_i \mid P_j \rangle = \frac{1}{i+\frac{1}{2}} \mathbb{1}_{i=j}$. So we define q_i to be a rescaled version of the standard Legendre polynomial:

$$q_i(a) := \sqrt{i + \frac{1}{2}} P_i(a), \tag{46}$$

so that

$$\langle q_i \mid q_j \rangle = \mathbb{1}_{i=j}. \tag{47}$$

Polynomial q_i has degree i , and the q_i 's form an orthonormal basis for polynomial functions.

We can now state our assumption.

Assumption 2 (Utility function as drawn from a random distribution on polynomial basis) *We*

³⁵More generally we have $P_i(a) = \frac{1}{2^i i!} \frac{d^i}{da^i} [(a^2 - 1)^i]$ by Rodrigues' formula.

decompose the true utility function $u(a)$ as:

$$u(a) = \sum_{i=-1}^{\infty} f_i Q_i(a) \quad (48)$$

where $Q_{-1}(a) \equiv 1$ and for $i \geq 0$, $Q_i(a) = \int_0^a q_i(y) dy$, where $q_i(y)$ is the i -th normalized Legendre polynomial (46). We assume that the subset I such that coefficients $\{f_i\}_{i \in I}$ are nonzero is finite, and that the f_i for $i \in I$ are i.i.d. and follow a $N(0, \sigma_u^2)$ distribution. Also assume $\sigma_{f_{-1}}^2 = \chi^2 \sigma_u^2$.

We note that, in the limit where all coefficients are non-zero, we get the “Wiener” case.

Result We prove a more general proposition, which allows for heteroskedastic priors, i.e. different $\sigma_{f_i}^2$ depending on i , and admits the homoskedastic priors of Assumption 2 as a special case.

Proposition 12 Suppose that coefficients f_i are drawn from the Gaussian $N(0, \sigma_{f_i}^2)$, and jointly Gaussian and uncorrelated. Then, the posterior $\mathbb{E}[u(a) | \mathbf{s}] := \mathbb{E}[u(a) | (s(y))_{y \in [-1, 1]}]$ is:

$$\mathbb{E}[u(a) | \mathbf{s}] = \sum_{i=-1}^{\infty} \mathbb{E}[f_i | \mathbf{s}] Q_i(a)$$

where, for $i \geq 1$

$$\begin{aligned} \mathbb{E}[f_i | \mathbf{s}] &= \lambda_i \langle q_i | d\mathbf{s} \rangle = \lambda_i \int_{a=-1}^1 q_i(a) ds(a) \\ \lambda_i &= 1 / (1 + \sigma^2 / \sigma_{f_i}^2) \end{aligned}$$

while $\mathbb{E}[f_{-1} | s] = \lambda_{-1} s(0)$ with $\lambda_{-1} = 1 / (1 + \chi^2 \sigma^2 / \sigma_{f_{-1}}^2)$. This implies that the average posterior is:

$$\bar{u}(a) := \mathbb{E}[[u(a) | \mathbf{s}] | f] = \sum_{i=-1}^{\infty} \lambda_i f_i Q_i(a). \quad (49)$$

In particular, take the case of flat (or “homoskedastic”) priors of Assumption 2, and call $\lambda = 1 / (1 + \frac{\sigma^2}{\sigma_u^2})$. Then,

$$\bar{u}(a) = \lambda u(a) \quad (50)$$

i.e. we obtain uniform dampening.

Proof of Proposition 12 Suppose that we have a function $u(a)$, and we observe, as in (30),

$$s(a) = u(a) + \sigma W(a) + \chi \sigma \eta_0 \quad (51)$$

where $W(a)$ is a Brownian motion and $\tilde{\eta}_0 = \chi\sigma\eta_0$ is a Gaussian variable of mean zero. Differentiate:

$$\begin{aligned} ds(a) &= u'(a) da + \sigma dW(a) \\ u'(a) &= \sum_{j=-1}^{\infty} f_j Q'_j(a) = \sum_{j=0}^{\infty} f_j q_j(a). \end{aligned}$$

Hence:

$$ds(a) = \sum_{j=0}^{\infty} f_j q_j(a) da + \sigma dW(a).$$

The agent wants to infer u given s , i.e. f given ds (we consider the intercept $u(0)$ at the end). Multiplying the previous equation by $q_i(a)$ and integrating between -1 and 1 gives:

$$\begin{aligned} S_i &:= \langle q_i \mid ds \rangle \\ &= \sum_j f_j \langle q_i \mid q_j \rangle + \sigma \langle q_i \mid dW \rangle \\ &= f_i + \sigma \langle q_i \mid dW \rangle \end{aligned} \tag{52}$$

because of (47).

Hence we can write the signal $S_i := \langle q_i \mid ds \rangle$ as

$$S_i = f_i + \sigma \varepsilon_i \tag{53}$$

with $\varepsilon_i := \langle q_i \mid dW \rangle = \int_{-1}^1 q_i(a) dW_a$ satisfies $\mathbb{E}[\varepsilon_i] = 0$. In addition, signal noises are uncorrelated:

$$\mathbb{E}[\varepsilon_i \varepsilon_j] = \mathbb{E} \left[\left(\int q_i(a) dW_a \right) \left(\int q_j(a) dW_a \right) \right] = \int q_i(a) q_j(a) da = \mathbb{1}_{i=j}$$

Hence, the signal-extraction problem $\mathbb{E}[f_i \mid \mathbf{s}]$ is quite simple, as only S_i is informative about f_i : $\mathbb{E}[f_i \mid \mathbf{s}] = \mathbb{E}[f_i \mid S_i]$. Given (53),

$$\mathbb{E}[f_i \mid \mathbf{s}] = \lambda_i S_i \tag{54}$$

$$\lambda_i = 1 / (1 + \sigma^2 / \sigma_{f_i}^2). \tag{55}$$

Hence, we have

$$\begin{aligned}\mathbb{E}[u'(a) | \mathbf{s}] &= \sum_{i=0}^{\infty} \mathbb{E}[f_i | \mathbf{s}] q_i(a) \\ &= \sum_{i=0}^{\infty} \mathbb{E}[f_i | \mathbf{s}] Q'_i(a)\end{aligned}$$

We next study the intercept in (51), $u(0)$. Given $s(0) = u(0) + \chi\sigma\eta_0$ and $u(0) = f_{-1}$,

$$\mathbb{E}[u(0) | \mathbf{s}] = \mathbb{E}[f_{-1} | s(0)] = \lambda_{-1}s(0) = \lambda_{-1}S_{-1}$$

where $S_{-1} := s(0)$ and $\lambda_{-1} = 1 / \left(1 + \chi^2\sigma^2/\sigma_{f_{-1}}^2\right)$. Integrating,

$$\begin{aligned}\mathbb{E}[u(a) | s] &= \mathbb{E}[u(0) | s] + \mathbb{E}\left[\int_0^a u'(b) db | s\right] \\ &= \lambda_{-1}S_{-1} + \sum_{i=0}^{\infty} \lambda_i S_i Q_i(a) = \sum_{i=-1}^{\infty} \lambda_i S_i Q_i(a).\end{aligned}$$

In addition, the average perception is:

$$\begin{aligned}\bar{u}(a) &:= \mathbb{E}[[u(a) | \mathbf{s}] | \mathbf{u}] = \sum_{i=-1}^{\infty} \lambda_i \mathbb{E}[S_i | f] Q_i(a) \\ &= \sum_{i=-1}^{\infty} \lambda_i f_i Q_i(a)\end{aligned}\tag{56}$$

If we assume a “flat” prior of Assumption 2, where $\sigma_{f_i}^2$ is independent of i (if $\sigma_{f_i}^2 > 0$), we have for $i \geq 0$

$$\lambda_i = \lambda = \frac{1}{1 + \frac{\sigma^2}{\sigma_{f_i}^2}} = \frac{1}{1 + \frac{\sigma^2}{\sigma_u^2}}.$$

Furthermore, as $\sigma_{f_{-1}}^2 = \chi^2\sigma_u^2$,

$$\lambda_{-1} = \frac{1}{1 + \frac{\chi^2\sigma^2}{\sigma_{f_{-1}}^2}} = \lambda.$$

Hence, $\lambda_i = \lambda$ for all $i \geq -1$, and (56) implies:

$$\bar{u}(a) = \sum_{i=-1}^{\infty} \lambda_i f_i Q_i(a) = \lambda \sum_{i=-1}^{\infty} f_i Q_i(a) = \lambda u(a).$$