Online Appendix for "Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance"

Xavier Gabaix

November 4 2011

This online appendix contains some complements to the paper: extension to many factors; the pricing of "equity strips;" details on the Epstein-Zin extension of the model; details on the simulations.

VII. Theory Complements

VII.A. The Model with Many Factors

The paper derived the main economics of the variable severity of disasters model. It relied on one nominal and one real risk factor. This section shows how the model readily extends to many factors, including those with high- and low-frequency predictability of dividend growth or inflation.

VII.A.1. Extensions for Stocks

Variable-Trend Growth Rate of Dividends It is easy to add a predictable growth-rate trend to the stock's dividend. Postulate:

$$\frac{D_{i,t+1}}{D_{it}} = e^{g_{iD}} \left(1+\varepsilon_{t+1}^{D}\right)\left(1+\widehat{g}_{it}\right) \times \begin{cases} 1 & \text{if there is no disaster at } t+1 \\ F_{it} & \text{if there is a disaster at } t+1 \end{cases},$$

where $\widehat{g}_{it}$ is the deviation of the growth rate from trend and follows an LG-twisted process: $\mathbb{E}_t \widehat{g}_{i,t+1} = \frac{1+H_{i*}}{(1+\widehat{g}_{it})(1+H_{it})} e^{-\phi_g} \widehat{g}_{it}$. Calling $\widehat{h}_{it} = (H_{it} - H_{i*})(1+\widehat{g}_{it}) / (1+H_{i*})$, postulate that

$$\mathbb{E}_t \widehat{h}_{i,t+1} = \frac{1+H_{i*}}{(1+\widehat{g}_{it})(1+H_{it})} e^{-\phi_H} \widehat{h}_{it}.$$

Proposition 11 *(Stock price with time-varying risk premia and time-varying growth rate of dividends) The price of stock i in the model with stochastic resilience $\widehat{H}_{it}$ and stochastic growth rate of dividend $\widehat{g}_{it}$ is in the limit of small time intervals:*

$$P_{it} = \frac{D_{it}}{\delta_i} \left(1 + \frac{\widehat{H}_{it}}{\delta_i + \phi_H} + \frac{\widehat{g}_{it}}{\delta_i + \phi_g}\right). \tag{47}$$

The expected return on the stock, conditional on no disaster, is still $r_{it}^{e} = \delta - h_{i*} - \widehat{H}_{it}$.

Proof. $M_t D_{it}\left(1, \widehat{h}_{it}, \widehat{g}_{it}\right)$ is an LG process, so the P/D ratio obtains. ■

Eq. 47 nests the three main sources of variation of stock prices in a simple and natural way. Stock prices can increase because the level of dividends (D_{it}) increases, because the expected future growth rate of dividends ($\widehat{g}_{it}$) increases, or because the equity premium decreases ($\widehat{H}_{it}$). The growth and discount factors ($\widehat{g}_{it}$, $\widehat{H}_{it}$) enter linearly, weighted by their duration (e.g., $1/\left(\delta_i + \phi_H\right)$) which depends on the speed of mean reversion of each process (ϕ_H, ϕ_g) and the effective discount rate δ_i. The price is independent of the correlation between the instantaneous innovations in $\widehat{g}_{it}$ and $\widehat{H}_{it}$, as is typical of LG processes.

Stocks: Many Factors To have several factors for the growth rate and the discount factor, postulate: $\widehat{g}_{it} = \sum_{k=1}^{N_g} \widehat{g}_{ikt}$, $\widehat{h}_{it} = \sum_{k=1}^{N_H} \widehat{h}_{ikt}$, $\mathbb{E}_t \widehat{g}_{ik,t+1} = \frac{1+H_{i*}}{(1+\widehat{g}_{it})(1+H_{it})} e^{-\phi_{g,k}} \widehat{g}_{ik,t}$, and $\mathbb{E}_t \widehat{h}_{ik,t+1} = \frac{1+H_{i*}}{(1+\widehat{g}_{it})(1+H_{it})} e^{-\phi_{H,k}} \widehat{h}_{ik,t}$. For instance, the growth rates could correspond to different frequencies, i.e., a long-run frequency (for low $\phi_{g,k}$) and a business-cycle frequency (for a high $\phi_{g,k}$).

Proposition 12 *(Stock prices with time-varying risk premium and time-varying growth rates of dividends with an arbitrary number of factors) The price of stock i in the model with stochastic resilience $\widehat{h}_{it} = \sum_{k=1}^{N_H} \widehat{h}_{k,t}$ and stochastic growth rate of dividend $\widehat{g}_{it} = \sum_{k=1}^{N_g} \widehat{g}_{k,t}$ is in the limit of small time intervals:*

$$P_{it} = \frac{D_{it}}{\delta_i}\left(1 + \sum_{k=1}^{N_H} \frac{\widehat{h}_{ikt}}{\delta_i + \phi_{H,k}} + \sum_{k=1}^{N_g} \frac{\widehat{g}_{ikt}}{\delta_i + \phi_{g,k}}\right). \tag{48}$$

The expected return on the stock, conditional on no disasters, is still: $r^e_{it} = \delta - h_{i*} - \widehat{h}_{it}$.

Proof. $M_t D_{it}\left(1, \widehat{h}_{i1t}, ..., \widehat{h}_{iN_k}, \widehat{g}_{i1,t}, ..., \widehat{g}_{iN_g,t}\right)$ is an LG process, so the P/D ratio obtains. ■

Formula (48) is very versatile, and could be applied to a large number of cases.

VII.A.2. Extensions for Bonds

Bonds: Variation in the Short-Term Real Rate To highlight the role of risk premia, the real short-term interest rate is held constant in the baseline model. We can make it variable, e.g., as follows. Postulate that consumption follows $C_t = \widetilde{C}_t C^*_t$, where C^*_t follows the process seen so far (Eq. 1) and $\widetilde{C}_t$ captures a deviation of consumption from trend. We could take $\widetilde{C}_{t+1}/\widetilde{C}_t = e^{g_t}$, where g_t would follow an AR(1). In the LG spirit, let us take a twisted version thereof, $\widetilde{C}_{t+1}/\widetilde{C}_t = (1 - R_t)^{-1/\gamma}$, where R_t follows an LG process, $\mathbb{E}_t\left[R_{t+1}\right] = e^{-\phi_R} R_t / \left(1 + R_t\right)$ with innovations in R_t that are uncorrelated with disasters and innovations in inflation variables I_t and J_t (the two processes would be similar for $g_t \simeq R_t/\gamma$). When the consumption growth rate is high, R_t is high. The pricing kernel is $M_t = \widetilde{C}_t^{-\gamma} M^*_t$, where M^*_t is given in Eq. 2.

Proposition 13 *(Bond prices in the extended model) The bond price in the extended model is:*

$$Z_{\$t}(T) = Z^*_{\$t}(T)\left(1 - \frac{1-e^{-\phi_R T}}{1-e^{-\phi_R}} R_t\right),$$

where $Z^*_{\$t}(T)$ is the bond price derived earlier in Theorem 2.

Proof. Because $\widetilde{C}_t^{-\gamma}(1, R_t)$ is an LG process, we have: $\mathbb{E}_t\left[\frac{\widetilde{C}_{t+T}^{-\gamma}}{\widetilde{C}_t^{-\gamma}}\right] = 1 - \frac{1-e^{-\phi_R T}}{1-e^{-\phi_R}} R_t$. With Q_t the real value of one dollar, the nominal bond price is: $Z_{\$t} = \mathbb{E}_t\left[\frac{M_{t+T}Q_{t+T}}{M_t Q_{t+T}}\right] = \mathbb{E}_t\left[\frac{M^*_t Q_{t+T}}{M^*_t Q_{t+T}}\right]\mathbb{E}_t\left[\frac{\widetilde{C}_{t+T}^{-\gamma}}{\widetilde{C}_t^{-\gamma}}\right] = Z^*_{\$t}(T)\,\mathbb{E}_t\left[\frac{\widetilde{C}_{t+T}^{-\gamma}}{\widetilde{C}_t^{-\gamma}}\right]$. ■

In the continuous-time limit, the short-term rate is $r_t = \delta - H_\$ + I_t + R_t$. It now depends both on inflation I_t and on the consumption growth factor R_t.

Bonds: Many Factors The bond model admits more factors. For instance, assume that inflation is the sum of K components, $I_t = I_* + \sum_{k=1}^K \widehat{I}_{kt}$, which follow:

$$\widehat{I}_{k,t+1} = \frac{1-I_*}{1-I_t}\cdot\left(\rho_k \widehat{I}_{kt} + 1_{\{\text{Disaster at } t+1\}}\left(j_{k*} + \widehat{J}_{kt}\right)\right) + \varepsilon_{k,t+1} \text{ for } k = 1, ..., K,$$

with $\widehat{J}_{k,t+1} = \frac{1-I_*}{1-I_t}\rho_{kj}\widehat{J}_t + \varepsilon^j_{t+1}$ and $\left(\varepsilon_{1t}, ..., \varepsilon_{Kt}, \varepsilon^j_{1t}, ..., \varepsilon^j_{Kt}\right)$ mean 0 independently of whether or not there is a disaster at t. The k-th component mean-reverts with an autocorrelation ρ_k, which allows us to model inflation as the sum of fast and slow components. If there is a disaster, the k-th component of inflation jumps by an amount $\widehat{J}_{k,t}$ which will mean-revert quickly if ρ_k is small. I state the bond price in the simple case with no average increase in inflation: $\forall k, j_{k*} = 0$.

Proposition 14 *(Bond prices with several factors) In the bond model with K factors, the bond price is:*

$$Z_{\$t}(T) = \left(e^{-\delta}(1+H_\$)(1-I_*)\right)^T (1 - \sum_{k=1}^K \frac{1-\rho_k^T}{1-\rho_k}\frac{\widehat{I}_{kt}}{1-I_*} - \sum_{k=1}^K \frac{\frac{1-\rho_k^T}{1-\rho_k} - \frac{1-\rho_{kj}^T}{1-\rho_{kj}}}{\rho_k - \rho_{kj}} \frac{p_t \mathbb{E}_t\left[B_{t+1}^{-\gamma}\right]\widehat{J}_{kt}}{1-I_*}). \quad (49)$$

The bond price decreases with each component of inflation, and more persistent components have greater impact.

To sum up, the framework makes it easy to model a variety of multi-factor features of stocks and bonds.

VII.B. *Equity Strips*

An "equity strip" of maturity T is the claim at time t on the dividend paid at time $t+T$. Equity strips may allow the study of the maturity structure of equity risk premia.[33] In that sense, they are the equity analogue of the yield curve (Binsbergen, Brandt, and Koijen forth.)

Fortunately, it is easy to calculate properties of an equity strip of maturity T on stock i, $P_{it}(T) = \mathbb{E}\left[M_{t+T}D_{it+T}/M_t\right]$. Recall that $M_t D_{it}\left(1, \widehat{H}_{it}\right)$ is an LG process with (I use the continuous-time limit here for simplicity) continuous-time generator $\omega = \begin{pmatrix} \delta_i & -1 \\ 0 & \delta_i + \phi_H \end{pmatrix}$, where $\delta_i = \delta - H_{i*} - g_D$. So, by Theorem 3 in Gabaix (2009),

$$P_{it}(T) = D_{it}e^{-\delta_i T}\left(1 + \frac{1 - e^{-\phi_H T}}{\phi_H}\widehat{H}_{it}\right), \tag{50}$$

while the stock price is $P_{it} = \frac{D_{it}}{\delta_i}\left(1 + \frac{\widehat{H}_{it}}{\delta_i + \phi_H}\right) = \int_0^\infty P_{it}(T)\, dT$.

The expected return on the strip (conditioning on no disaster, which is the common benchmark in this nascent literature and which I think is the relevant comparison for post-Great-Depression data) is, by the same reasoning as in Proposition 1,

$$r_{it}^e(T) = \delta - H_{it}. \tag{51}$$

Hence, the expected return is the same across maturities T. The reason is that strips of all maturities are (in this simplest version of the disaster model) exposed to the same risk in a disaster: the value of all equity strips will be reduced by a factor F_{it}.

Let us now calculate the return volatility (also conditional on no disasters). Taking a linearization, the return is:

$$r_{it}(T) = \varepsilon_{it}^D + \frac{1 - e^{-\phi_H T}}{\phi_H}\varepsilon_{it}^H,$$

where ε_{it}^D is the innovation to $\ln D_{it}$ and ε_{it}^H is the innovation to $\widehat{H}_{it}$. So the volatility of the return is:

$$\sigma_{it}(T) = \sqrt{\sigma_D^2 + \left(\frac{1 - e^{-\phi_H T}}{\phi_H}\right)^2 \sigma_H^2}. \tag{52}$$

We see that the return volatility is increasing with maturity. The reason is the same as for bonds. Longer-maturity bonds have more volatility, because they have a higher duration: shocks are compounded over a long maturity.

[33] I thank Ralph Koijen for prompting me to think about the price of equity strips.

Finally, the Sharpe ratio is:

$$SR_{it}(T) = \frac{\delta - H_{it}}{\sqrt{\sigma_D^2 + \left(\frac{1-e^{-\phi_H T}}{\phi_H}\right)^2 \sigma_H^2}},$$

which is decreasing with maturity. Indeed, as the limit $\sigma_D \to 0$, the Sharpe ratio of short-maturity strips becomes infinite. The reason is that in a disaster model, an asset can have zero "normal-times volatility," but a positive risk premium, which reflects the latent disaster risk.

Adding a stochastic growth rate can be useful to model growth stocks, as in Section VII.A. Then, the price of the strip is:

$$P_{it}(T) = D_{it} e^{-\delta_i T} \left(1 + \frac{1-e^{-\phi_H T}}{\phi_H} \widehat{H}_{it} + \frac{1-e^{-\phi_g T}}{\phi_g} \widehat{g}_{it}\right),$$

so the volatility is:

$$\sigma_{it}(T) = \sqrt{\sigma_D^2 + \left(\frac{1-e^{-\phi_H T}}{\phi_H}\right)^2 \sigma_H^2 + \left(\frac{1-e^{-\phi_g T}}{\phi_g}\right)^2 \sigma_g^2}.$$

The expected return is still $\delta - H_{it}$, so the Sharpe Ratio is:

$$SR_{it}(T) = \frac{\delta - H_{it}}{\sqrt{\sigma_D^2 + \left(\frac{1-e^{-\phi_H T}}{\phi_H}\right)^2 \sigma_H^2 + \left(\frac{1-e^{-\phi_g T}}{\phi_g}\right)^2 \sigma_g^2}}.$$

Thus, the prediction seems robust to adding a stochastic growth rate.

Given that the stock price is

$$P_{it}(T) = \frac{D_{it}}{\delta_i} \left(1 + \frac{\widehat{H}_{it}}{\delta_i + \phi_H} + \frac{\widehat{g}_{it}}{\delta_i + \phi_g}\right),$$

the stock price volatility is:

$$\sigma_r = \sqrt{\sigma_D^2 + \frac{\sigma_H^2}{(\delta_i + \phi_H)^2} + \frac{\sigma_g^2}{(\delta_i + \phi_g)^2}}.$$

VII.C. Complement to the Cochrane-Piazzesi Part

Here I complement Section IV.C on Cochrane-Piazzesi. The baseline model has only 2 factors. Let us consider a small deviation with 3 factors. Add an extra factor to the model (e.g., an extra component to inflation, as in Section VII.A.2), r_t, that mean-reverts at a low speed ψ_r. Then, along the multi-factor model of Section VII.A.2, we would have, to the leading order:

$$f_t(T) = F(T) + e^{-\psi_r T} r_t + e^{-\psi_I T} I_t + \Lambda(T)\pi_t.$$

This is like in the paper, but with an extra $e^{-\psi_r T} r_t$ term. Now, find the weights w_T on 3 maturities, $\{a, b, a+2b\}$ (e.g., $\{1 \text{ yr}, 3 \text{ yr}, 5 \text{ yr}\}$), such that the combination $\sum_{T=a,a+b,a+2b} w_T f_t(T)$ is the bond risk premium up to a constant:

$$\pi_T = \text{constant} + \sum_{T=a,a+b,a+2b} w_T f_t(T).$$

The combination is now generically unique (it is unique if ψ_I, ψ_j, and ψ_r are different), and it can be obtained by making sure the right-hand side of the above equation has zero weight on r_t and I_t. The weight $w = (w_a, w_{a+b}, w_{a+2b})$ is:

$$w = \frac{\left(\psi_j - \psi_i\right) e^{(a+2b)\psi_j}}{\left(e^{\psi_j b} - e^{\psi_i b}\right)\left(e^{\psi_j b} - e^{\psi_r b}\right)} \left(-1, e^{b\psi_i} + e^{b\psi_r}, -e^{b(\psi_i+\psi_r)}\right)'.$$

In the case of interest ($\psi_j > \psi_i$, ψ_r, so that the bond risk premium mean-reverts faster than inflation), the weights have signs $(-,+,-)$ and are, thus, tent-shaped. In the limit ψ_I, $\psi_r \to 0$ (used in the rest of the paper), we have

$$w = k(-1, 2, -1)'$$

for a constant $k = \frac{\psi_j e^{a\psi_j}}{\left(1-e^{-b\psi_j}\right)^2}$, so that $k \sim \frac{1}{b^2\psi_j}$ when ψ_j is small. This gives the "estimation-free" combination proposed in (31).

VII.D. *Additional Derivations*

Proof of Theorem 2 in the continuous-time case In the proof, I normalize $I_* = 0$. I will show that $M_t Q_t (1, I_t, \pi_t)$ is an LG process. I calculate its three LG continuous-time moments. Successively,

$$\mathbb{E}_t\left[\frac{d(M_t Q_t)}{M_t Q_t}\right]/dt = \underbrace{-(\delta + I_t)}_{\text{No disaster term}} + p_t \underbrace{\left(\mathbb{E}_t\left[F^{\$}_{t+\varepsilon} B^{-\gamma}_{t+\varepsilon}\right] - 1\right)}_{\text{Disaster term}} = -\delta + H_{\$} - I_t$$

$$
\begin{aligned}
\mathbb{E}_t\left[\frac{d\left(M_tQ_tI_t\right)}{M_tQ_t}\right]/dt &= \underbrace{-\left(\delta+I_t\right)I_t+\mathbb{E}_t^{ND}\left[dI_t/dt\right]}_{\text{No disaster term}}+p_t\underbrace{\left(\mathbb{E}_tB_t^{-\gamma}F_t^{\$}\mathbb{E}_t\left[I_{t+dt}\right]-i_t\right)}_{\text{Disaster term}} \\
&= -\left(\delta+I_t\right)I_t-\left(\phi_I-I_t\right)I_t+p_t\left(\mathbb{E}_tB_t^{-\gamma}F_t^{\$}\left(I_t+J_*+\widehat{J}_t\right)-I_t\right) \\
&= p_t\mathbb{E}_tB_t^{-\gamma}F_t^{\$}J_*-\left(\delta+\phi_I-H_{\$}\right)I_t+p_t\mathbb{E}_tB_t^{-\gamma}F_t^{\$}\widehat{J}_t \\
&= \kappa\left(\phi_I-\kappa\right)-\left(\delta+\phi_I-H_{\$}\right)I_t+\pi_t,
\end{aligned}
$$

as I defined $p_t\mathbb{E}_tB_t^{-\gamma}F_t^{\$}J_*=\kappa\left(\phi_I-\kappa\right)$ and $\pi_t=p_tB_t^{-\gamma}F\widehat{J}_t$. Finally:

$$
\begin{aligned}
\mathbb{E}_t\frac{d\left(M_tQ_t\pi_t\right)}{M_tQ_t}/dt &= \underbrace{-\left(\delta+I_t\right)\pi_t+\mathbb{E}_td\pi_t/dt}_{\text{No disaster term}}+p_t\underbrace{\left(\mathbb{E}_tB_t^{-\gamma}F_t^{\$}\pi_t-\pi_t\right)}_{\text{Disaster term}} \\
&= -\left(\delta-H_{\$}+I_t\right)\pi_t-\left(\phi_J-I_t\right)\pi_t=-\left(\delta-H_{\$}+\phi_J\right)\pi_t.
\end{aligned}
$$

I conclude that $M_tQ_t\left(1,I_t,\pi_t\right)'$ is an LG process with generator

$$
\begin{pmatrix}
\delta-H_{\$} & 1 & 0 \\
-\kappa\left(\phi_I-\kappa\right) & \delta-H_{\$}+\phi_I & -1 \\
0 & 0 & \delta-H_{\$}+\phi_J
\end{pmatrix}.
$$

When $\kappa=0$ (inflation has no bias during disaster), the proof can directly go to its conclusion, (53). When $\kappa\neq0$, one more step is needed. Define $\widetilde{i}_t=I_t-\kappa$. Then, the process $M_tQ_t\left(1,\widetilde{i}_t,\pi_t\right)$ is LG with generator:

$$
\omega_1=\begin{pmatrix}
\delta-H_{\$}+\kappa & 1 & 0 \\
0 & \delta-H_{\$}+\phi_I-\kappa & -1 \\
0 & 0 & \delta-H_{\$}+\phi_J
\end{pmatrix}=\left(\delta-H_{\$}+\kappa\right)I_3+\begin{pmatrix}
0 & 1 & 0 \\
0 & \psi_I & -1 \\
0 & 0 & \psi_J
\end{pmatrix},
$$

where $\psi_I=\phi_I-2\kappa$, $\psi_J=\phi_J-\kappa$, and I_3 the 3×3 identity matrix. Theorem 3 in Gabaix (2009) provides the bond price:

$$
\begin{aligned}
Z_{\$t}\left(T\right) &= \left(1,0,0\right)e^{-\left(\delta-H_{\$}+\kappa\right)T}\exp\left(-\begin{pmatrix}
0 & 1 & 0 \\
0 & \psi_I & -1 \\
0 & 0 & \psi_J
\end{pmatrix}T\right)\left(1,\widetilde{i}_t,\pi_t\right)' \\
&= e^{-\left(\delta-H_{\$}+\kappa\right)T}\left(1-\frac{1-e^{-\psi_IT}}{\psi_I}\left(I_t-I_{**}\right)-\frac{\frac{1-e^{-\psi_IT}}{\psi_I}-\frac{1-e^{-\psi_JT}}{\psi_I}}{\psi_J-\psi_I}\pi_t\right).
\end{aligned}
\tag{53}
$$

Proof of the Epstein-Zin part Here I provide the proofs for the Epstein-Zin part of the model, including the exact results in the Appendix.

Stochastic discount factor

We verify that the equilibrium price of a consumption claim is indeed consistent with (36); we show that it leads to a consistent process. First, the return on a consumption claim is:

$$
\begin{aligned}
dR_t &= \frac{dP_t}{P_t} + \frac{C_t}{P_t}dt = \frac{dC_t}{C_t} + \frac{d(P_t/C_t)}{P_t/C_t} + \frac{C_t}{P_t}dt \\
&= \frac{dC_t}{C_t} + \frac{\frac{\chi}{\delta_c+\phi_c}dk_t}{1+\frac{\chi k_t}{\delta_c+\phi_c}} + \left(\frac{\delta_c}{1+\frac{\chi k_t}{\delta_c+\phi_c}}dt - \delta_c dt\right) + \delta_c dt \\
&= \frac{dC_t}{C_t} + \delta_c dt + \frac{\chi dk_t}{\delta_c+\phi_c+\chi k_t} + \frac{-\delta_c \chi k_t dt}{\delta_c+\phi_c+\chi k_t} \\
&= \frac{dC_t}{C_t} + \delta_c dt + \chi \frac{-\left(\delta_c+\phi_c+\chi k_t\right) k_t dt + \left(\delta_c+\phi_c+\chi k_t\right)\left[\left(\chi-1\right) w^2\left(k_t\right) dt + w\left(k_t\right) dz_t\right]}{\delta_c+\phi_c+\chi k_t} \\
dR_t &= \frac{dC_t}{C_t} + \left(\delta_c - \chi k_t + \chi\left(\chi-1\right) w^2\left(k_t\right)\right) dt + \chi w\left(k_t\right) dz_t. \qquad (54)
\end{aligned}
$$

The SDF satisfies $M_{t+dt}/M_t = e^{-\rho/\chi dt}\left(C_{t+dt}/C_t\right)^{-1/(\chi\psi)}\left(1+dR_t\right)^{1/\chi-1}$ in semi-discrete-time notation (the fully-continuous-time machinery of Duffie-Epstein 1992 would give the same result), so that in continuous time, if there is no disaster:

$$
\begin{aligned}
\frac{dM_t}{M_t}_{|ND} &= -\frac{\rho}{\chi}dt - \frac{1}{\chi\psi}g_c dt + \left[\left(1+\left(g_c+\delta_c-\chi k_t+\chi\left(\chi-1\right)w^2\left(k_t\right)\right)dt + \chi w\left(k_t\right)dz_t\right)^{1/\chi-1} - 1\right] \\
&= -\left(\frac{\rho}{\chi}+\frac{1}{\chi\psi}g_c\right)dt + \left(\frac{1}{\chi}-1\right)\left(\left(\delta-\chi H_{c*}-\chi k_t+\chi\left(\chi-1\right)w^2\left(k_t\right)\right)dt+\chi w\left(k_t\right)dz_t\right) \\
&\quad + \left(\frac{1}{\chi}-1\right)\left(\frac{1}{\chi}-2\right)\frac{\chi^2}{2}w\left(k_t\right)^2 dt \\
&= -\frac{\delta}{\chi}dt + \left(\frac{1}{\chi}-1\right)\left(\delta-\chi H_{c*}\right)dt + \left(\chi-1\right)\left[\left(k_t+\frac{w^2\left(k_t\right)}{2}\right)dt - w\left(k_t\right)dz_t\right] \\
&= \left(-\delta+\left(\chi-1\right)H_{c*}\right)dt + \left(\chi-1\right)\left[\left(k_t+\frac{w^2\left(k_t\right)}{2}\right)dt - w\left(k_t\right)dz_t\right].
\end{aligned}
$$

If there is a disaster, M_t is multiplied by $B_t^{-1/(\chi\psi)+1/\chi-1} = B_t^{-\gamma}$. So,

$$
\begin{aligned}
\frac{dM_t}{M_t} &= \frac{dM_t}{M_t}_{|ND} + \left(B_t^{-\gamma}-1\right)d\mathcal{J}_t \\
&= \left(-\delta+\left(\chi-1\right)H_{c*}\right)dt + \left(\chi-1\right)\left[\left(k_t+\frac{w^2\left(k_t\right)}{2}\right)dt - w\left(k_t\right)dz_t\right],
\end{aligned}
$$

i.e., (37). Note that here are elsewhere, to be a purist, we could write $\frac{dM_t}{M_{t^-}}$ rather than $\frac{dM_t}{M_t}$. Given the context, I think that there's no ambiguity and so little gain in augmenting the notational burden with a t^-.

Price of a consumption claim

We now need to verify that a consumption claim satisfies (36). We will show that $M_t C_t\left(1,k_t\right)$

is an LG process. From (37), we calculate:

$$\frac{d\left(M_tC_t\right)}{M_tC_t} = -\delta dt + (\chi - 1)\left(\left(H_{c*} + k_t + \frac{w^2\left(k_t\right)}{2}\right)dt - w\left(k_t\right)dz_t\right) + g_c dt + \left(B_t^{1-\gamma} - 1\right)d\mathcal{J}_t.$$

The first LG moment is:

$$\begin{aligned}
\mathbb{E}_t\left[\frac{d\left(M_tC_t\right)}{M_tC_t}\right]/dt &= -\delta + (\chi - 1)\left(H_{c*} + k_t + \frac{w^2\left(k_t\right)}{2}\right) + g_c + p_t\mathbb{E}\left[B_t^{1-\gamma} - 1\right] \\
&= -\delta + g_c + (\chi - 1)\left(H_{c*} + k_t + \frac{w^2\left(k_t\right)}{2}\right) + H_{ct} \\
&= -\delta + g_c + (\chi - 1)\left(H_{c*} + k_t + \frac{w^2\left(k_t\right)}{2}\right) + H_{c*} + k_t + \frac{1-\chi}{2}w^2\left(k_t\right) \\
&= -\delta + g_c + \chi H_{c*} + \chi k_t \\
&= -\delta_c + \chi k_t. \qquad (55)
\end{aligned}$$

The second LG moment is:

$$\begin{aligned}
\mathbb{E}_t\left[\frac{d\left(M_tC_tk_t\right)}{M_tC_t}\right]/dt &= \mathbb{E}_t\left[\frac{d\left(M_tC_t\right)}{M_tC_t}\right]/dt\cdot k_t + \mathbb{E}\left[dk_t\right]/dt + \left\langle\frac{dM_t}{M_t}, dk_t\right\rangle \\
&= \left(-\delta_c + \chi k_t\right)k_t - \left(\phi_c + \chi k_t\right)k_t + \left(\delta_c + \phi_c + \chi k_t\right)(\chi - 1)\,w^2\left(k_t\right) \\
&\quad + (1-\chi)\left(\delta_c + \phi_c + \chi k_t\right)w^2\left(k_t\right) \\
&= -\left(\delta_c + \phi_c\right)k_t.
\end{aligned}$$

The results on LG processes (Gabaix 2009, Theorem 4) give $P_t = C_t\left(1 + \frac{\chi k_t}{\delta_c + \phi_c}\right)$, i.e., (36).

A general stock

We will show that $M_tD_{it}\left(1, \widehat{H}_{it}^{EZ}\right)$ is an LG process.

$$\begin{aligned}
\frac{d\left(M_tD_{it}\right)}{M_tD_{it}} &= -\delta dt + (\chi - 1)\left(\left(H_{c*} + k_t + \frac{w^2\left(k_t\right)}{2}\right)dt - w\left(k_t\right)dz_t\right) + g_{iD}dt \\
&\quad + \left\langle\frac{dM_t}{M_t}, \frac{dD_{it}}{D_{it}}\right\rangle dt + \left(B_t^{-\gamma}F_{it} - 1\right)d\mathcal{J}_t \\
\mathbb{E}_t\left[\frac{d\left(M_tD_{it}\right)}{M_tD_{it}}\right]/dt &= -\delta + (\chi - 1)\left(H_{c*} + k_t + \frac{w^2\left(k_t\right)}{2}\right) + g_{iD} + \left\langle\frac{dM_t}{M_t}, \frac{dD_{it}}{D_{it}}\right\rangle dt + H_{it} \\
&= -\delta + H_{it}^{EZ} = -\delta + H_{i*}^{EZ} + \widehat{H}_{it}^{EZ} = -\delta_i + \widehat{H}_{it}^{EZ}
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}_t\left[\frac{d\left(M_t D_{it}\widehat{H}_{it}^{EZ}\right)}{M_t D_{it}}\right]/dt &= \left(-\delta_i+\widehat{H}_{it}^{EZ}\right)\widehat{H}_{it}^{EZ}+\left\langle\frac{dM_t}{M_t}+\frac{dD_{it}}{D_{it}},d\widehat{H}_{it}^{EZ}\right\rangle+\mathbb{E}\left[d\widehat{H}_{it}^{EZ}\right]/dt \\
&= \left(-\delta_i+\widehat{H}_{it}^{EZ}\right)\widehat{H}_{it}^{EZ}-\left(\phi_H+\widehat{H}_{it}^{EZ}\right)\widehat{H}_{it}^{EZ} \\
&= -\left(\delta_i+\phi_H\right)\widehat{H}_{it}^{EZ}.
\end{aligned}
$$

By the LG results, this proves (34).

The stock price is higher if the dividends covary with the price kernel in normal times, or if resilience innovations covary with it.[34]

VII.E. On the Calibration of the Epstein-Zin Model

This section adds details on the calibration of the EZ model.

How big are the second-order terms? The Appendix presents an exactly solved EZ model, and the main text presents its first-order impact in order to demonstrate the main economics. Here I study how big the "second-order terms" are. We shall see that they are less than 1% or 0.1% of the quantities of interest, so that the model in the main text is a very accurate rendition of the more complex model in the Appendix.

To see this, I use an average volatility of the disaster probability $\sigma_p = 0.4\%$ (in annualized units) from the main text (it is an average, because it has to go to 0 when p is close to 0). We have $\sigma_H = \left(B^{1-\gamma}-1\right)\sigma_p$, so for the k_t process in the Appendix, the volatility of k is:

$$
\sigma_k = \left(B^{1-\gamma}-1\right)\sigma_p = 1.0\%.
$$

Recall that for an AR(1) process $dx_t = -\phi x_t dt + \sigma dz_t$, the dispersion of x_t is $v_x := \mathbb{E}\left[x_t^2\right]^{1/2} = \sigma/\sqrt{2\phi}$. To the leading term, this holds for a twisted process (Gabaix 2009), so the typical size of k (its dispersion) is:

$$
v_k = \frac{\sigma_k}{\sqrt{2\phi_c}} = 2.0\%.
$$

We can now evaluate how small the "second-order terms" are. For instance, in M_{t+1}/M_t and in r_t, the main text approximates $H_{ct}+\frac{\chi w^2(k_t)}{2}$ by H_{ct}, neglecting the second-order term (see Eq. 32 vs. Eq. 37, and the expressions for r_{ft} after those equations). How big is that second-order term $\frac{\chi w^2(k_t)}{2}$ compared to a first-order term H_{ct} (or $\widehat{H}_{ct}$)? It is only a fraction 0.0008 of the leading term,

[34] The latter effect can be seen in two ways: first, by just calculating the expected return as $-cov\left(\frac{dM_t}{M_t},dr_t\right)/dt$, or otherwise by observing that, by (39), a positive average moment $\langle\frac{dM_t}{M_t}+\frac{dD_{it}}{D_{it}},dN_{it}^{EZ}\rangle_{ND}$ leads to a negative mean for $\widehat{H}_{it}^{EZ}$; as H_{it} is given in the partition H_{it} between $\widehat{H}_{it}^{EZ}$ and H_{i*}, this leads to a higher value for H_{i*}^{EZ}, which implies a higher price P_{it}.

H_{ct} $(= \frac{\chi w^2(k_t)}{2} / v_{H_c}$ with $v_{H_c} = v_k)$, using a "ratio of average absolute values" as a metric. Likewise, the approximation in the interest rate only amounts to the same fraction 0.0008.

Similarly, for the process of H_{ct}, the "second-order term," which is in the Appendix but omitted in the main text, is $(\delta_c + \phi_c + \chi k_t)(\chi - 1) w^2(k_t)$ (equation 35). It is very small compared to the first-order term, $\phi_c k_t$: only a fraction 0.008 $(= |\frac{(\delta_c+\phi_c+\chi k_t)(\chi-1)\sigma_k^2}{\phi_c v_k}|)$.

Calibration of the rate of time preference The interest rate in the EZ model is given by $r_{ft}dt = -\mathbb{E}[dM_t/M_t]$, i.e., to the first order:

$$r_{ft} = \rho^{EZ} + \frac{g_c}{\psi} - p_t \mathbb{E}^D \left[B_t^{-\gamma} \left(1 + (\chi - 1) B_t\right) - \chi \right].$$

Targeting as in the power utility model a mean short-term rate of 1% yields $\rho^{EZ} = 4.8\%$.

Quantitative impact of variable p_t vs variable F_{it} The variance given in the paper comes from (56), with $F_{i*} = \overline{B}$ as in the paper:

$$\widehat{H}_{it}^{EZ} = \chi \widehat{p}_t \overline{B}^{1-\gamma} + p_* \overline{B}^{-\gamma} \widehat{F}_{it},$$

so that

$$\begin{aligned} \sigma_H^{EZ} &= \sqrt{\left(\overline{B}^{1-\gamma}\right)^2 \sigma_p^2 + \left(p_* \overline{B}^{-\gamma}\right)^2 \sigma_F^2} \\ &= 1.01 \sigma_H^{CRRA}, \end{aligned}$$

where $\sigma_H^{CRRA} = p_* \overline{B}^{-\gamma} \sigma_F$ is the volatility of resilience in the power utility calibration.

The "normal-times" volatility of the SDF can be derived from $\varepsilon_{t+1}^M = (1 - \chi) \frac{\varepsilon_{c,t+1}^H}{\delta_c + \phi_H}$ and $H_{Ct}^{EZ} = p_t \mathbb{E}_t^D \left[B_{t+1}^{1-\gamma} - 1 \right]$, which implies $\varepsilon_{c,t+1}^H = \varepsilon_{t+1}^p \left(\overline{B}^{1-\gamma} - 1 \right)$, so that:

$$\sigma\left(\varepsilon_{t+1}^M\right) = |(1 - \chi)| \frac{\left(\overline{B}^{1-\gamma} - 1\right) \sigma_p}{\delta_c + \phi_H} = 6.4\%.$$

One could imagine several other useful calibrations (Wachter 2009 presents a different one) that might be a combination of the above, e.g., with moderate variability of p_t and variability of F_{it}.

Simulation of the Epstein-Zin model with both p_t and F_{it}^{EZ} variable The problem is the following: consider the process for H_{it}^{EZ} defined in the main paper (Eq. 39). How is it derived from movements of both p_t and F_{it}?

A simple way to solve this is as follows. For simplicity, I take the baseline case where $\left\langle \frac{dM_t}{M_t}, \frac{dD_{it}}{D_{it}} \right\rangle^{ND} =$

0. Call

$$\overline{H}_{it}^{EZ} = p_t \mathbb{E}_t^D \left[B_{t+1}^{-\gamma} \left(F_{it} + (\chi - 1) B_t \right) - \chi \right]$$

the "principal part" of H_{it}^{EZ}, so by (38), $H_{it}^{EZ} = \overline{H}_{it}^{EZ} + (\chi - 1) \frac{\chi w^2(k_t)}{2}$. As $w^2(k_t)$ is independent of the asset, we want to simulate $\overline{H}_{it}^{EZ}$. Call $\widehat{p}_t = p_t - p_*$ and $\widehat{F}_{it} = F_{it} - F_{i*}$ the deviations from trends. Heuristically, $\widehat{\overline{H}}_{it}^{EZ} \simeq \overline{H}_{i*}^{EZ} + \frac{\partial \overline{H}_{it}^{EZ}}{\partial p_t} \widehat{p}_t + \frac{\partial \overline{H}_{it}^{EZ}}{\partial F_{it}} \widehat{F}_{it}$. Thus, given a process for p_t and a process for $\overline{H}_{it}^{EZ}$, we can define a "subordinate" process for $\widehat{F}_{it}$ such that the following equation holds exactly:

$$\widehat{\overline{H}}_{it}^{EZ} = \widehat{p}_t \left(\mathbb{E}_t^D \left[B_{t+1}^{-\gamma} \left(F_{it} + (\chi - 1) B_t \right) - \chi \right] \right)_* + \left(p_t \mathbb{E}_t^D \left[B_t^{-\gamma} \right] \right)_* \widehat{F}_{it}, \tag{56}$$

where $\left(p_t \mathbb{E}_t^D \left[B_t^{-\gamma} \right] \right)_*$ is the constant part of $p_t \mathbb{E}_t^D \left[B_t^{-\gamma} \right]$. The above gives a well-defined process for $\widehat{F}_{it}$.

As in the rest of the paper, we first define the movement of the "high-level" variables (e.g., resiliences), and then define the appropriate movement for the "low-level" variables, e.g., $\widehat{F}_{it}$.

VIII. Complements to the Part on Options

In the options section, the paper assumes that the distribution of stock returns, conditional on no disasters, is lognormal. This section shows how to ensure that by means of a more general result.

Position of the problem. Suppose we are given a random variable (r.v.) Y with CDF G_Y and a desired CDF G_Z with support in $(0, \infty)$ such that the associated r.v. satisfies: $\mathbb{E}[Y] = \mathbb{E}[Z]$. We want to define a r.v. X such that X is uncorrelated with Y and XY has the CDF G_Z.

Can we find such a r.v. X? We provide sufficient conditions for this and a construction of X.

Lemma 3 *(Constructive way to "noise up" a r.v. to obtain a desired distribution) Consider two positive independent random variables (r.v.) Y and Z_0, with CDFs, respectively, G_Y and G_Z, where $\mathbb{E}[Y] = \mathbb{E}[Z_0]$. Define the r.v. $Z_1 = G_Z^{-1}(G_Y(Y))$. Suppose that*

$$Q \equiv \mathbb{E}\left[\frac{Z_1}{Y} \right] \leq 1. \tag{57}$$

Then, define:

$$\lambda = \frac{1 - Q}{\mathbb{E}[Y] \mathbb{E}[1/Y] - Q} \in [0, 1]$$

and the r.v.:

$$X = \begin{cases} Z_0 / Y & \text{with probability } \lambda \\ Z_1 / Y & \text{with probability } 1 - \lambda \end{cases},$$

where the probability is taken independently of Y, Z_0. Then, XY has CDF G_Z, and X is uncorrelated with Y and $\mathbb{E}[X] = 1$.

Proof. Recall that if $u \sim U[0,1]$ r.v., then the r.v. $G_Z^{-1}(u)$ has CDF G_Z. Also, $G_Y(Y) \sim U[0,1]$. Given this, it is straightforward to see that Z_1 and $Z = XY$ have CDF G_Z. Also, $\mathbb{E}[Y]\mathbb{E}[1/Y] \geq 1$, so that $\lambda = (1-Q)/(\mathbb{E}[Y]\mathbb{E}[1/Y] - Q) \leq 1$. Finally,

$$
\begin{aligned}
\mathbb{E}[X] &= \lambda E[Z_0/Y] + (1-\lambda)\mathbb{E}[Z_1/Y] \\
&= \lambda E[Z_0]\mathbb{E}[1/Y] + (1-\lambda)Q = \lambda E[Y]\mathbb{E}[1/Y] + (1-\lambda)Q = 1.
\end{aligned}
$$

Thus, $cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[Z_0] - 1 \cdot \mathbb{E}[Y] = 0$. ■

We can also show that if Y's distribution is very tight around 1, then $Q \leq 1$. So, if Y has very little dispersion, then it is possible to find a required X.

The intuition for the condition $Q \leq 1$ is the following. If Y and Z are independent, then $\mathbb{E}[Z/Y] = \mathbb{E}[Z]\mathbb{E}[1/Y] \geq 1$. So Y and Z need to be positively correlated or "affiliated." The case of maximum affiliation is $Z_1 = G_Z^{-1}(G_Y(Y))$. Thus, if $Q \equiv \mathbb{E}[Z_1/Y]$ is ≤ 1, then there is a case of "intermediate affiliation" such that $\mathbb{E}[Z/Y] = 1$. The above constructed X is one such example.

Condition (57) means roughly that Z is more dispersed than Y. To illustrate this, consider the case where Y and Z are lognormal: $Y = \exp(-s^2/2 + sm)$, $Z = \exp(-\sigma^2/2 + \sigma n)$, where m, n are standard normal and σ, s are non-negative. Then, $Z_1 = \exp(-\sigma^2/2 + \sigma m)$ (as it has the right distribution and is perfectly affiliated with Y) and:

$$
\begin{aligned}
Q &= \mathbb{E}\left[\frac{Z_1}{Y}\right] = \mathbb{E}\left[\exp\left(-\sigma^2/2 + \sigma m - \left(-s^2/2 + sm\right)\right)\right] = \exp\left(-\sigma^2/2 + s^2/2 + (\sigma - s)^2/2\right) \\
&= \exp(s(s-\sigma)).
\end{aligned}
$$

Hence, $Q \leq 1$ iff $s \leq \sigma$, which means that Z is more dispersed than Y.

I suspect that the condition (57) is close to necessary for the existence of X. At the same time, there are many ways to construct the r.v. X.

Application to options In the paper, define $Y = \frac{a+b\left(\frac{e^{-\phi_H}\widehat{H}_{it}}{1+e^{-h^*}\widehat{H}_{it}} + \varepsilon^H_{i,t+1}\right)}{a+b\widehat{H}_{it}} e^{g_{iD}}$, $Z_0 = e^{\mu + \sigma u_{t+1} - \sigma^2/2}$, and $X = 1 + \varepsilon^D_{t+1}$. Calling ND the probabilities conditional on no disasters, we have $\mathbb{E}^{ND}[Z_0] = \mathbb{E}^{ND}[Y]$. The paper requires $\mathbb{E}^{ND}[X] = 1$ and $cov^{ND}(X,Y) = 0$. So, we just apply Lemma 3, which yields a r.v. X. We then define $1 + \varepsilon^D_{t+1} = X$ if there is no disaster, and $1 + \varepsilon^D_{t+1} = 1$ if there is a disaster. Then, by construction, the distribution of P_{t+1}/P_t is as announced in (22).

IX. Details for the Simulations

Units are annual. I simulate at the monthly frequency, $\Delta t = 1/12$. When the regressions have lower frequencies (e.g., annual), I time-aggregate. I use the notation $x^+ = \max(0,x)$.

Bonds Core inflation $I_t = I_* + \widehat{I}_t$ and the bond premium π_t are expressed in annual units. They follow:

$$\begin{aligned}\widehat{I}_{t+1} &= \frac{e^{-\phi_I \Delta t}\widehat{I}_t}{1-\widehat{I}_t\Delta t} + K_{t+1}\sigma_I\sqrt{\Delta t}\varepsilon^I_{t+1} \\ \pi_{t+1} &= \frac{e^{-\phi_J \Delta t}\pi_t}{1-\widehat{I}_t\Delta t} + K_{t+1}\sigma_\pi\sqrt{\Delta t}\varepsilon^\pi_{t+1}\end{aligned}$$

$$K_{t+1} = \min\left(1, \left(1 - \frac{\widehat{I}^+_t}{\psi_I} - \frac{\pi^+_t}{\psi_I\psi_J}\right)^+ /0.1\right),$$

and ε^I_{t+1}, ε^π_{t+1} are independent random variables with uniform distribution $\left[-\sqrt{3}, \sqrt{3}\right]$, so that their variance is 1. As distributions do not matter in LG processes, any other bounded distribution (e.g., a truncated Gaussian) could replace the uniform one. The "killing" term K_{t+1} ensures that the volatility goes to 0 at the boundary of the region $1 - \frac{\widehat{I}^+_t}{\psi_t} - \frac{\pi^+_t}{\psi_I\psi_J} \geq 0$, which safeguards that bond prices remain positive.

I then simulate 1,000 years of data, and report the results from this time series. I also check that it does not make too much of a difference if I simulate many time series with 50 years of data. Bond prices are as given in Theorem 2.

Stocks The stock's normalized resilience $\widehat{h}_t = e^{-h_*}\widehat{H}_{it}$ is expressed in annual units. I define:

$$\begin{aligned}G(x) &= \frac{e^{-\phi_H \Delta t}x}{1+x\Delta t} \\ \sigma(x) &= \sqrt{2K}\left(1 - x/\widehat{h}_{\min}\right)\left(1 - x/\widehat{h}_{\max}\right) \\ v(x) &= \min\left(\sigma(x)\sqrt{3\Delta t}, \widehat{h}_{\max} - F\left(\widehat{h}_t\right), F\left(\widehat{h}_t\right) - \widehat{h}_{\min}\right)\end{aligned}$$

and simulate:[35]

$$\widehat{h}_{t+1} = G\left(\widehat{h}_t\right) + v\left(\widehat{h}_t\right)\varepsilon^H_{i,t+1}, \tag{58}$$

with $\varepsilon^H_{i,t+1}$ i.i.d. with uniform distribution $[-1, 1]$, so that their standard deviation is $1/\sqrt{3}$.

The function $G(x)$ is simply the twisted AR(1) process simulated with time intervals Δt.

The function $\sigma(x)$ is the volatility discussed below.

In $v(x)$, the term $\sigma(x)\sqrt{3\Delta t}$ ensures that the standard deviation of shocks is $\sigma(x)$ away from the boundaries. The terms $\widehat{h}_{\max} - G\left(\widehat{h}_t\right)$ and $G\left(\widehat{h}_t\right) - \widehat{h}_{\min}$ prevent $\widehat{h}_t$ from hitting the boundary in discrete time.

[35] Checking that the process satisfies (5) is trivial: multiplying (5) by e^{-h_*} shows that it is equivalent to $\mathbb{E}\widehat{h}_{t+1} = \frac{1+H_{i*}}{1+H_{i*}+\widehat{H}_{it}}e^{-\phi_H}\widehat{h}_t = \frac{e^{-\phi_H}}{1+\widehat{h}_t}e^{-\phi_H}\widehat{h}_t = G\left(\widehat{h}_t\right)$. Other verifications follow similar lines.

The dividend follows a geometric random walk: $\ln D_{i,t+1} = \ln D_{it} + g_{iD}\Delta t + \sigma_D \sqrt{\Delta t} \varepsilon^D_{i,t+1}$, where $\varepsilon^D_{t+1} \sim U\left[-\sqrt{3}, \sqrt{3}\right]$ is i.i.d. with unit variance.

The stock price is as in Theorem 1.

A variance process vanishing at boundaries Suppose X_t is an LG-twisted process centered at 0 and $dX_t = -(\phi + X_t) X_t dt + \sigma(X_t) dW_t$, where W_t is a standard Brownian motion. Assume that the support of X_t is some $[X_{\min}, X_{\max}]$, with $-\phi < X_{\min} < 0 < X_{\max}$. The following variance process makes that possible:

$$\sigma^2(X) = 2K(1 - X/X_{\min})^2 (1 - X/X_{\max})^2, \tag{59}$$

with $K > 0$. For the steady-state distribution to have a unimodal shape, it is useful to take $K = 0.2 \cdot \phi |X_{\min}| X_{\max}$ (the value used in the simulations), or a lower value.

The average variance of X_t is $\overline{\sigma}^2_X = \mathbb{E}[\sigma^2(X_t)] = \int_{X_{\min}}^{X_{\max}} \sigma(x)^2 p(x) dx$, where $p(x)$ is the steady-state distribution of X_t. It can be calculated via the Forward Kolmogorov equation, which yields $d \ln p(X)/dX = 2X(\phi + X)/\sigma^2(X) - d \ln \sigma^2(X)/dX$. Numerical simulations show that the volatility of the process is fairly well approximated by $\overline{\sigma}_X \simeq K^{1/2}\xi$, with $\xi = 1.3$. Also, the standard deviation of X's steady-state distribution is well approximated by $(K/\phi)^{1/2}$. Asset prices often require that one analyze the standard deviation of expressions like $\ln(1 + aX_t)$. Numerical analysis shows that the Taylor expansion approximation is a good one. The average volatility of $\ln(1 + aX_t)$ is $\overline{\sigma}_{\ln(1+aX_t)} \simeq aK^{1/2}\xi$, which numerical simulations also show to be a good approximation.

When the process is not centered at 0, one simply centers the values. For instance, in the calibration, the recovery rate of a stock F_{it} has support $[F_{\min}, F_{\max}] = [0, 1]$ centered around F_{i*}. In the simulation, for simplicity I assume that conditionally on a disaster F_{it} has a point mass. The probability and intensity of disasters (p and B) are constant. Formula $H_{it} = p(\mathbb{E}[B^{-\gamma}] F_{it} - 1)$ gives the associated $H_{\min}$, $H_{\max}$, and H_{i*}. The associated centered process is $X_t = \widehat{H}_{it} = H_{it} - H_{i*}$.